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BIJECTION BETWEEN INDEXED MONOMIALS AND STANDARD BITABLEAUX

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0. Introduction

Let X be an $m(1)$ by $m(2)$ matrix whose entries X_{ij} are independent indeterminates over a field K. Let a be a *bivector* of length p bounded by $m=(m(1), m(2))$, i.e. let a be a pair of increasing sequences of positive integers

$$a(1,p)>\cdots>a(1,2)>a(1,1)\,||\,a(2,1)<a(2,2)<\cdots<a(2,p)$$

such that $a(k,p)\leqslant m(k)$ for $k=1,2$. Also let V be a nonnegative integer. By $\operatorname{mon}(2, m, p, a, V)$ we denote the exponent system of a certain finite set of "indexed" monomials in the $m(1)m(2)$ indeterminates X_{ij} determined by a and V, and by $\operatorname{stab}(2, m, p, a, V)$ we denote a certain finite set of standard bitableaux determined by m, a, V. Detailed definitions of the sets $\operatorname{mon}(2, m, p, a, V)$ and $\operatorname{stab}(2, m, p, a, V)$ will be given in a moment.

In Theorem (9.9) of Abhyankar [3], by enumeration it was proved that the two sets $\operatorname{mon}(2, m, p, a, V)$ and $\operatorname{stab}(2, m, p, a, V)$ have the same cardinality, and in Remark (9.10) of Abhyankar [3] it was suggested that a bijective proof of this be found. One aim of this paper is to give such a bijective proof by setting up a one-to-one correspondence between the sets $\operatorname{mon}(2, m, p, a, V)$ and $\operatorname{stab}(2, m, p, a, V)$. As a consequence of this correspondence we shall also give a bijective proof of the Straightening Law of Doubilet–Rota–Stein [8]. The said correspondence is obtained by modifying the RSK correspondence, i.e. the correspondence given by Robinson [12], Schensted [13] and Knuth [9]. The RSK correspondence is based on a procedure of inserting a positive integer in a standard unitableau. The said procedure is also described in the Sorting and Searching Volume of Knuth's book on *The Art of Computer Programming* [10].

Now a (Young) tableau is a tabular arrangement of positive integers such as the one depicted on the next page.

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1	3	4	5	9
2	5	6	7	9
3	5	6	8	
3	9			
7				
8				

In a tableau it is required to have a strict increase along the rows. The tableau is said to be *standard* if, as in the above example, the row lengths are nonincreasing and there is a nondecrease along the columns.

The above example is an example of a *unitableau.* A tableau may have two similar sides, i.e. two sides of the same shape but with possibly different entries, and then it is called a *bitableau.* For instance the following is an example of a standard bitableau:

7	6	5	3	2		1	3	4	5	9
8	7	5	4	2		2	5	6	7	9
	8	7	5	4		3	5	6	8	
			6	5		3	9			
				5		7				
				7		8				

The number of entries on each side is called the *area* of the bitableau. For instance the area of the above tableau is 18. The left hand side of a bitableau may be called its first side and the right hand side may be called its second side. A bitableau is said to be *bounded* by the pair of positive integers $m = (m(1), m(2))$, if for $k = 1, 2$, all its entries on the kth side are less equal $m(k)$; it follows that each row of a bitableau bounded by m is a bivector bounded by m. For instance the above bitableau is bounded by any pair of integers $(m(1), m(2))$ with $m(1) \geqslant 8$ and $m(2) \geqslant 9$. The length of each side of the top row is called the *length* of the standard bitableau, and the number of rows in it is called its *depth*. For example the length of the above bitableau is 5, and its depth is 6. A standard bitableau is said to be *predominated* by the bivector a if the bitableau obtained by putting a on top of the given bitableau is again standard. For instance the above bitableau is predominated by the bivector depicted below.

7	6	4	3	1		1	2	4	5	6

The set of all standard bitableaux of area V which are bounded by m is denoted by stib$(2, m, V)$. Moreover, the set of all standard bitableaux of area V which are bounded by m and whose length is at most p is denoted by stab$[2, m, p, V]$, and the set of all standard bitableaux of area V which are bounded by m and predominated by a is denoted by stab$(2, m, p, a, V)$. Now a bivector bounded by m is precisely the code for indicating a minor of X. For instance the above bivector indicates the 5 by 5 minor of X whose row numbers are 1, 3, 4, 6, 7 and whose column numbers are 1, 2, 4, 5, 6. Upon taking the product of the minors which correspond to the various rows of a bitableau S bounded by m we get a monomial in the minors of X which we denote by mom(X, S).

Let cub$(2, m)$ be the set of all pairs of positive integers $y = (y(1), y(2))$ such that $y(1) \leq m(1)$ and $y(2) \leq m(2)$, and let mon$(2, m)$ be the set of all maps of cub$(2, m)$ into nonnegative integers. For every t in mon$(2, m)$ let supp(t) be the set of all y in cub$(2, m)$ for which $t(y) \neq 0$, and let abs$(t) = \sum t(y)$ with summation over cub$(2, m)$, and let X^t be the monomial of degree abs(t) in the indeterminates X_{ij} given by $X^t = \prod X^{t(y)}_{y(1),y(2)}$ with product over cub$(2, m)$. Let mon$[[2, m, V]]$ be the set of all t in mon$(2, m)$ such that abs$(t) = V$. For any subset Y of cub(2, m) let ind(Y) denote the largest nonnegative integer j for which there exist elements $y_1, \ldots, y_j$ in Y such that $y_i(k) < y_{i+1}(k)$ for $k = 1,2$ and $i = 1, \ldots, j-1$; we call ind(Y) the *index* of Y. Let mon$(2, m, p)$ be the set of all t in mon$(2, m)$ such that ind(supp$(t)) \leq p$, and let mon$(2, m, p, a)$ be the set of all t in mon$(2, m, p)$ such that ind$(t_{ki}) < i$ for $k = 1, 2$ and $i = 1, \ldots, p$ where t_{ki} is the set of all y in supp(t) with $y(k) < a(k, i)$. Finally let mon$[2, m, p, V]$ be the set of all t in mon$(2, m, p)$ such that abs$(t) = V$, and let mon$(2, m, p, a, V)$ be the set of all t in mon$(2, m, p, a)$ such that abs$(t) = V$.

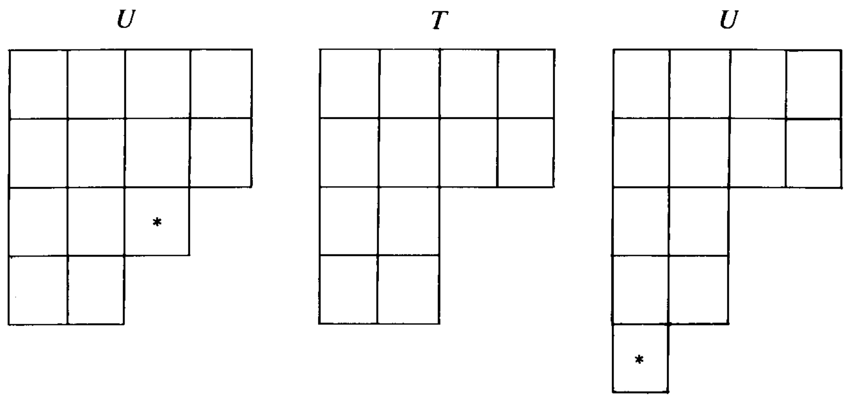
As noted above, in Theorem (9.9) of Abhyankar [3], by enumeration it was proved that the two sets mon$(2, m, p, a, V)$ and stab$(2, m, p, a, V)$ have the same cardinality. The said enumerative proof was effected by using certain recurrence properties of these two sets and by making a series of transformations of determinantal polynomials in binomial coefficients. Using Theorem (9.9), in Theorems (20.4) and (20.10) of Abhyankar [3] it was shown that both of these two sets serve as bases of the residue class ring $K[X]/I(p, a)$ where $K[X]$ is the polynomial ring in the $m(1)m(2)$ indeterminates X_{ij}, and $I(p, a)$ is a certain determinantal ideal in $K[X]$. In greater detail, let $I(p, a)$ be the homogeneous ideal in $K[X]$ generated by the sets $G(p+1)$, $G(p, a, 1, 1), \ldots, G(p, a, 1, p)$, $G(p, a, 2, 1), \ldots, G(p, a, 2, p)$, where $G(p+1)$ is the set of all $p+1$ by $p+1$ minors of X, and $G(p, a, 1, i)$ (resp: $G(p, a, 2, i)$) is the set of all i by i minors of X whose row numbers are $< a(1, i)$ (resp: column numbers are $< a(2, i)$). Then

$$(X^t)_{t \in \mathrm{mon}(2,m,p,a,V)} \text{ and } (\mathrm{mom}(X, S))_{S \in \mathrm{stab}(2,m,p,a,V)}$$

are K-bases of the Vth homogeneous component of $K[X]/I(p, a)$. It may be noted that Abhyankar [1] is a precursor of Abhyankar [3]. Moreover, a brief survey of Abhyankar [3] may be found in Abhyankar [2].

As a special case of Theorems (9.9), (20.4) and (20.10), by taking $a(1, i) = i = a(2, i)$ for $i = 1, \ldots, p$, we get Theorems (9.10*) and (20.5*) which say that the two sets $\mathrm{mon}[2, m, p, V]$ and $\mathrm{stab}[2, m, p, V]$ have the same cardinality, and hence each of them provides a basis of the residue class ring $K[X]/I(p+1)$ where $I(p+1)$ is the homogeneous ideal in $K[X]$ generated by $G(p+1)$; more precisely, $(X^t)_{t \in \mathrm{mon}[2,m,p,V]}$ and $(\mathrm{mom}(X, S))_{S \in \mathrm{stab}[2,m,p,V]}$ are K-bases of the Vth homogeneous component of $K[X]/I(p+1)$. As another special case of the above Theorem (9.9), in Theorem (9.11) of Abhyankar [3] it was noted that the two sets $\mathrm{mon}[[2, m, V]]$ and $\mathrm{stib}(2, m, V)$ have the same cardinality. In Theorem (16.11) of Abhyankar [3] it was shown that $\mathrm{mom}(X, S)_{S \in \mathrm{stib}(2,m,V)}$ generates the Vth homogeneous component of $K[X]$, and in Theorem (20.3) of Abhyankar [3] it was noted that Theorems (9.11) and (16.11) yield the Straightening Law which says that $\mathrm{mom}(X, S)_{S \in \mathrm{stib}(2,m,V)}$ is a K-basis of the Vth homogeneous component of $K[X]$. Alternatively, the Straightening Law also follows from Theorem (9.11) of Abhyankar [3] and Theorem (3.6.2) of Abhyankar–Ghorpade [4] which gives a direct proof of the linear independence of $\mathrm{mom}(X, S)_{S \in \mathrm{stib}(2,m,V)}$. For some other proofs of the Straightening Law see DeConcini–Eisenbud–Procesi [6] and Desarmenian–Kung–Rota [7], and as a general reference for Young tableaux see Kung [11]. As said before, in this paper we shall give bijective proofs of the above Theorems (9.9), (9.10*) and (9.11).

In Definition (5.1) of Section 5 we shall describe the dual RSK procedure of *inserting* a positive integer z in a standard unitableau T. Briefly, if z is bigger than everybody in the first row (of T) then z is inserted at the end of the first row; otherwise z is less equal somebody in the first row and we let z *bump* the first such element $x(1, 2)$; if $x(1, 2)$ is bigger than everybody in the second row then we insert $x(1, 2)$ at the end of the second row; otherwise $x(1, 2)$ is less equal somebody in the second row and we let $x(1, 2)$ bump the first such element $x(1, 3)$; if $x(1, 3)$ is bigger than everybody in the third row then we insert $x(1, 3)$ at the end of the third row; otherwise. . .; and so on. This procedure stops when a new slot is created, say in the sth row and vth column. The resulting unitableau U is again seen to be standard. It may also be noted that, as depicted below, the new slot is on the *periphery* of T, i.e. the length of the sth row of T is $v-1$, and, for every $e < s$, the length of the eth row of T is $\geqslant v$; equivalently, the length of the sth row of U is v, and, for every $e > s$, the length of the eth row of U is $<v$; for example, in the following figure, $*$ denotes the new slot and, for the same T, two possible U's are depicted.



Then in Definition (6.1) of Section 6 we shall describe the reverse procedure of going back from U to T by *deleting* the (s, v)th slot. Moreover, in Definition (5.2), starting with the empty unitableau, and successively inserting the elements of a finite sequence of positive integers w, we get a standard unitableau $R(w)$ which we call the *roinsertion* of w where *ro* stands for row. In Lemmas (5.3) and (6.4) we prove some basic properties of the insertion and deletion procedures which enable us to extend these procedures to bitableaux. Following Knuth [9], in Section 2 we introduce an *equivalence relation* in the set of all finite sequences of positive integers, and in Theorem (5.12) we show that two sequences are equivalent iff their roinsertions coincide. Given any standard unitableau T, in Section 2 we construct two sequences of positive integers $\mathrm{vas}(T)$ and $\mathrm{covas}(T)$ which we call the *vectorial associate* and the *covectorial associate* of T, and in Lemmas (5.8) and (5.9) we show that T is the roinsertion of $\mathrm{vas}(T)$ as well as $\mathrm{covas}(T)$; in other words, $\mathrm{vas}(T)$ and $\mathrm{covas}(T)$ are two canonical members of the equivalence class of sequences whose roinsertion is T.

To extend the above procedure of roinsertion to bitableaux, we fix $k = 1$ or 2 and we let $k' = 2$ or 1 respectively. Now given any t in $\mathrm{mon}(2, m)$, upon letting $r = \mathrm{abs}(t)$, in Definition (8.14) of Section 8 we define a pair of sequences of positive integers $w(k, 1), w(k, 2), \ldots, w(k, r)$ and $w(k', 1), w(k', 2), \ldots, w(k', r)$ such that $w(k', 1) \leqslant w(k', 2) \leqslant \cdots \leqslant w(k', r)$ and such that $w(k, i) \geqslant w(k, i+1)$ whenever $w(k', i) = w(k', i+1)$ and such that $X^t = \prod X_{w(1,i),w(2,i)}$ where the product is over $i = 1, \ldots, r$; we call this pair of sequences the *lexical associate* of t. In Definition (8.16) we define a standard bitableau $\mathrm{MR}_{k,m}(t)$ whose kth side is constructed by applying the above procedure of roinsertion to the sequence $w(k, 1), w(k, 2), \ldots, w(k, r)$. While doing this construction, as new empty slots are created on the k'th side, we fill them in by successive members of the sequence $w(k', 1), w(k', 2), \ldots, w(k', r)$. The standard bitableau $\mathrm{MR}_{k,m}(t)$ is called the *monomial roinsertion* of (k, m, t). In Definition (9.12) of Section 9 we give reverse procedure of *monomial rodeletion*. In Theorem (9.13) we show that this sets up a one-to-one correspondence between the sets $\mathrm{mon}(2, m)$ and $\mathrm{stab}(2, m)$, i.e. the resulting map $\mathrm{MR}_{k,m}: \mathrm{mon}(2, m) \to \mathrm{stab}(2, m)$ is bijective. Then in Theorems (9.14) to (9.16) we show that the said map induces bijective maps $\mathrm{mon}[[2, m, V]] \to \mathrm{stib}(2, m, V)$ and $\mathrm{mon}[2, m, p, V] \to \mathrm{stab}[2, m, p, V]$ and $\mathrm{mon}(2, m, p, a, V) \to \mathrm{stab}(2, m, p, a, V)$.

As a key to our treatment of roinsertion, in Section 2 we consider the lattice of multivectors, and in terms of it we define the *vectorial* GLB (=*greatest lower bound*) $\mathrm{veg}(w)$ of a multisequence of positive integers w, and we also consider the integer $\mathrm{inc}(w)$ which is the *length of the longest increasing subsequence* of w. In Theorem (5.7) we prove that, if w is a unisequence of positive integers, i.e. an ordinary finite sequence of positive integers, then $\mathrm{veg}(w)$ is the first row of the unitableau $R(w)$, and hence in particular the length of $R(w)$ equals $\mathrm{inc}(w)$; this last result is the original motivation of Schensted [13]. In Section 2, in an analogous manner, we also define $\mathrm{veg}(t)$ for any t in $\mathrm{mon}(2, m)$, and as a consequence of Theorem (5.7), in Theorem (8.17) we show that $\mathrm{veg}(t)$ is the first row of the bitableau $\mathrm{MR}_{k,m}(t)$, and hence in particular the length of $\mathrm{MR}_{k,m}(t)$ equals $\mathrm{ind}(\mathrm{supp}(t))$. Theorems (5.7) and (8.17) may be regarded as the main ingredients of this paper.

Actually, most of the analysis of roinsertion in this paper is done at the level of sequences instead of tableaux, and this helps to demystify the RSK correspondence. Thus, most of the work of Sections 5 and 6 is already done in Sections 3 and 4 where we deal with inserting a positive integer, or a finite sequence of positive integers, into a *univector* by which we mean an increasing finite sequence of positive integers. Similarly, most of the work of Sections 8 and 9 is already done in Section 7 where we deal with inserting a pair of integers, or a bisequence of positive integers, into a bivector. For example, the precursor of Definition (5.1) is Definition (3.1) of Section 3 where we describe the procedure of inserting a positive integer in a univector. This procedure described in Definition (3.1) is then the basic building block of the RSK correspondence and its modifications. Likewise, the precursor of Theorem (5.7) is Theorem (3.6) where we give an inductive construction of $\mathrm{veg}(w)$ and $\mathrm{inc}(w)$ for any finite sequence of positive integers w. So Theorem (3.6) may very well be regarded as the central theme of this paper.

In a forthcoming paper [5] we shall give a variation of the insertion procedure described in Definitions (5.1) and (5.2). This variation will be called *coinsertion* because it involves inserting along columns rather than rows. Given any finite sequence of positive integers w, the coinsertion procedure will produce a standard unitableau $C(w)$. Given any t in $\mathrm{mon}(2, m)$, by using coinsertion instead of roinsertion, we shall get a standard bitableaux $\mathrm{MC}_{k,m}(t)$ which we shall call the *monomial coinsertion* of (k, m, t). This will set up a one-to-one correspondence between certain finite sets comon$(2, m, p, a, V)$ and costab$(2, m, p, a, V)$ which are analogs of the sets mon$(2, m, p, a, V)$ and stab$(2, m, p, a, V)$. Following Knuth [9] we shall show that, for any finite sequence of positive integers w, we have $C(w) = R(w)$; this will enable us to give a complete characterization of the first column of $R(w)$; moreover, for any t in mon$(2, m)$, it will also enable us to give a partial characterization of the first column of $\mathrm{MR}_{k,m}(t)$.

1. Notation

As usual, by Z (resp: N, N^*) we denote the set of all integers (resp: nonnegative integers, positive integers). For any nonnegative integer p, by $Z(p)$ (resp: $N(p)$, $N^*(p)$) we denote the set of all sequences $n = n(i)_{1 \leq i \leq p}$ with $n(i)$ in Z (resp: in N, in N^*). Note that the equality of two sequences means the equality of their lengths and the equality of their corresponding components. For any A and B in Z, by $[A, B]$ we denote the set of all C in Z such that $A \leq C \leq B$. By card we denote cardinal number.

2. The lattice of multivectors and the concepts of standard tableau and index of a monomial

We shall now review some of the definitions about multivectors, tableaux and monomials from (2.2) to (2.4) of Abhyankar [3], and we shall add some more. So let there be fixed any $q \in N^*$.

Given any $p \in N$, by a *premultivector* a of *width* q and length p we mean a multisequence $a(k, i)_{1 \leq k \leq q, 1 \leq i \leq p}$ with $a(k, i) \in Z$; we say that a is *positive* if $a(k, i) \in N^*$ for all $k \in [1, q]$ and $i \in [1, p]$; by pre$[q, p]$ (resp: popre$[q, p]$) we denote the set of all premultivectors (resp: positive premultivectors) of width q and length p. By a *premultivector* of *width* q we mean a premultivector a of width q and length p for some $p \in N$, and we then put len$(a) = p$; by pre(q) (resp: popre(q)) we denote the set of all premultivectors (resp: positive premultivectors) of width q. For every $a \in$ popre(q) and $i \in [1, \text{len}(a)]$, by $a[i]$ we denote the unique member of $N^*(q)$ such that $a[i](k) = a(k, i)$ for all $k \in [1, q]$. By a *multivector* (resp: *comultivector*, *anticomultivector*) of *width* q we mean a positive premultivector a of width q such that $a(k, i) < a(k, i+1)$ (resp: $a(k, i) \leq a(k, i+1)$, $a(k, i) \geq a(k, i+1)$) for all $k \in [1, q]$ and $i \in [1, \text{len}(a) - 1]$; by vec$(q)$ (resp: covec(q), acovec(q)) we denote the set of all multivectors (resp: comultivectors, anticomultivectors) of width q, and given any $p \in N$, by vec$[q, p]$ (resp: covec$[q, p]$, acovec$[q, p]$) we denote the set of all multivectors (resp: comultivectors, anticomultivectors) of width q and length p, and by $\text{vec}_0(q)$ we denote the unique member of vec$[q, 0]$. Given any $a \in$ pre(q) and $m \in Z(q)$, we define $a \leq m$

to mean that $a(k, i) \leqslant m(k)$ for all $k \in [1, q]$ and $i \in [1, \operatorname{len}(a)]$, and we may express this by saying that a is *bounded* by m. Given any $m \in Z(q)$, by $\operatorname{pre}(q, m)$ (resp: $\operatorname{popre}(q, m)$, $\operatorname{vec}(q, m)$, $\operatorname{covec}(q, m)$, $\operatorname{acovec}(q, m)$) we denote the set of all a in $\operatorname{pre}(q)$ (resp: $\operatorname{popre}(q)$, $\operatorname{vec}(q)$, $\operatorname{covec}(q)$, $\operatorname{acovec}(q)$) such that $a \leqslant m$. Given any $m \in Z(q)$ and $p \in N$, by $\operatorname{pre}(q, m, p)$ (resp: $\operatorname{popre}(q, m, p)$, $\operatorname{vec}(q, m, p)$, $\operatorname{covec}(q, m, p)$, $\operatorname{acovec}(q, m, p)$) we denote the set of all a in $\operatorname{pre}[q, p]$ (resp: $\operatorname{popre}[q, p]$, $\operatorname{vec}[q, p]$, $\operatorname{covec}[q, p]$, $\operatorname{acovec}[q, p]$) such that $a \leqslant m$. By a *preunivector* (resp: *univector, counivector, anticounivector*) we mean a premultivector (resp: multivector, comultivector, anticomultivector) of width 1. By a *prebivector* (resp: *bivector, cobivector, anticobivector*) we mean a premultivector (resp: multivector, comultivector, anticomultivector) of width 2.

Given any $a \in \operatorname{pre}(q)$ and $k \in [1, q]$, we define $k(a) \in \operatorname{pre}[1, \operatorname{len}(a)]$ by putting $k(a)(1, i) = a(k, i)$ for all $i \in [1, \operatorname{len}(a)]$ and we note that $k(a)$ may be called the kth *side* of a. Given any $a \in \operatorname{pre}(q)$, we define $\operatorname{op}(a) \in \operatorname{pre}[q, \operatorname{len}(a)]$ by putting $\operatorname{op}(a)(k, i) = a(k, \operatorname{len}(a) + 1 - i)$ for all $k \in [1, q]$ and $i \in [1, \operatorname{len}(a)]$ and we note that $\operatorname{op}(a)$ may be called the *opposite* of a.

Given any a and a' in $\operatorname{pre}(q)$, we define $a \leqslant a'$ to mean that $\operatorname{len}(a) \geqslant \operatorname{len}(a')$ and $a(k, i) \leqslant a'(k, i)$ for all $k \in [1, q]$ and $i \in [1, \operatorname{len}(a')]$. This makes $\operatorname{pre}(q)$ into a *lattice*, i.e. a partially ordered set in which every nonempty finite subset Y has a (unique) GLB (=*greatest lower bound*) and a (unique) LUB (=*least upper bound*), i.e. there exists a unique element $\operatorname{GLB}(Y) \in \operatorname{pre}(q)$ such that

$$\operatorname{GLB}(Y) \leqslant y \text{ for all } y \in Y$$

and such that

$$[Y^* \in \operatorname{pre}(q) \text{ and } Y^* \leqslant y \text{ for all } y \in Y] \Rightarrow Y^* \leqslant \operatorname{GLB}(Y),$$

and there exists a unique element $\operatorname{LUB}(Y) \in \operatorname{pre}(q)$ such that

$$y \leqslant \operatorname{LUB}(Y) \text{ for all } y \in Y$$

and such that

$$[Y^* \in \operatorname{pre}(q) \text{ and } y \leqslant Y^* \text{ for all } y \in Y] \Rightarrow \operatorname{LUB}(Y) \leqslant Y^*.$$

Indeed, $\operatorname{GLB}(Y)$ can be characterized by saying that it is the unique element in $\operatorname{pre}(q)$ such that

$$\operatorname{len}(\operatorname{GLB}(Y)) = \max\{\operatorname{len}(y) : y \in Y\}$$

and for all $k \in [1, q]$ and $i \in [1, \operatorname{len}(\operatorname{GLB}(Y))]$ we have

$$\operatorname{GLB}(Y)(k, i) = \min\{y(k, i) : y \in Y \text{ with } \operatorname{len}(y) \geqslant i\}.$$

Similarly, $\operatorname{LUB}(Y)$ can be characterized by saying that it is the unique element in $\operatorname{pre}(q)$ such that

$$\operatorname{len}(\operatorname{LUB}(Y)) = \min\{\operatorname{len}(y) : y \in Y\}$$

and for all $k \in [1, q]$ and $i \in [1, \operatorname{len}(\operatorname{LUB}(Y))]$ we have

$$\operatorname{LUB}(Y)(k, i) = \max\{y(k, i) \colon y \in Y\}.$$

In view of these characterizations it follows that, for every $k \in [1, q]$ we have

$$k(\operatorname{GLB}(Y)) = \operatorname{GLB}(\{k(y) \colon y \in Y\}) \tag{2.1}$$

and

$$k(\operatorname{LUB}(Y)) = \operatorname{LUB}(\{k(y) \colon y \in Y\}). \tag{2.2}$$

We observe that if $Y \subset \operatorname{popre}(q)$ then clearly $\operatorname{GLB}(Y) \in \operatorname{popre}(q)$ and $\operatorname{LUB}(Y) \in \operatorname{popre}(q)$. *We also claim that if* $Y \subset \operatorname{vec}(q)$ *then* $\operatorname{GLB}(Y) \in \operatorname{vec}(q)$ *and* $\operatorname{LUB}(Y) \in \operatorname{vec}(q)$; to see this, given any $k \in [1, q]$ and $i < j$ in $[1, \operatorname{len}(\operatorname{GLB}(Y))]$, we can find $A \in Y$ such that $\operatorname{len}(A) \geqslant j$ and $\operatorname{GLB}(Y)(k, j) = A(k, j)$ and now we get

$$\operatorname{GLB}(Y)(k, i) \leqslant A(k, i) < A(k, j) = \operatorname{GLB}(Y)(k, j),$$

and similarly, given any $k \in [1, q]$ and $i < j$ in $[1, \operatorname{len}(\operatorname{LUB}(Y))]$, we can find $B \in Y$ such that $\operatorname{LUB}(Y)(k, i) = B(k, i)$ and now we get

$$\operatorname{LUB}(Y)(k, i) = B(k, i) < B(k, j) \leqslant \operatorname{LUB}(Y)(k, j).$$

Likewise *we claim that if* $Y \subset \operatorname{covec}(q)$ *then* $\operatorname{GLB}(Y) \in \operatorname{covec}(q)$ *and* $\operatorname{LUB}(Y) \in \operatorname{covec}(q)$; to see this, given any $k \in [1, q]$ and $i < j$ in $[1, \operatorname{len}(\operatorname{GLB}(Y))]$, we can find $A \in Y$ such that $\operatorname{len}(A) \geqslant j$ and $\operatorname{GLB}(Y)(k, j) = A(k, j)$ and now we get

$$\operatorname{GLB}(Y)(k, i) \leqslant A(k, i) \leqslant A(k, j) = \operatorname{GLB}(Y)(k, j),$$

and similarly, given any $k \in [1, q]$ and $i < j$ in $[1, \operatorname{len}(\operatorname{LUB}(Y))]$, we can find $B \in Y$ such that $\operatorname{LUB}(Y)(k, i) = B(k, i)$ and we now get

$$\operatorname{LUB}(Y)(k, i) = B(k, i) \leqslant B(k, j) \leqslant \operatorname{LUB}(Y)(k, j).$$

Given any $w \in \operatorname{pre}(q)$ and $S \subset [1, \operatorname{len}(w)]$, clearly there exists a unique increasing bijection $[1, \operatorname{card}(S)] \to S$, and we define $S[w] \in \operatorname{pre}[q, \operatorname{card}(S)]$ by saying that for all $k \in [1, q]$ and $i \in [1, \operatorname{card}(S)]$ we have

$$S[w](k, i) = w(k, \text{image of } i \text{ under the said increasing bijection})$$

and we remark that $S[w]$ may be called the *subsequence of w induced by S.*

Given any $w \in \operatorname{pre}(q)$, firstly we put

$$\operatorname{suve}(w) = \{S \subset [1, \operatorname{len}(w)] \colon S[w] \in \operatorname{vec}(q)\}$$

and we remark that a member of $\operatorname{suve}(w)$ may be called a *subvector* of w, and secondly we put

$$\operatorname{veg}(w) = \operatorname{GLB}(\{S[w] \colon S \in \operatorname{suve}(w)\})$$

and we note that

$$\operatorname{veg}(w) \in \operatorname{vec}(q)$$

and we remark that $\operatorname{veg}(w)$ may be called the *vectorial* GLB of w, and thirdly upon letting

$$\begin{cases} a = \text{veg}(w) \text{ and } p = \text{len}(a) \\ \text{and } W^*(i) = \{S \in \text{suve}(w)\colon \text{card}(S) \geqslant i\} \text{ for all } i \in N \\ \text{and } W(i) = \{S \in \text{suve}(w)\colon \text{card}(S) = i\} \text{ for all } i \in N \end{cases}$$

we note that

$$W^*(i) = \emptyset = W(i) \text{ for all } i \in N^* \setminus [1, p] \tag{2.3}$$

and

$$\begin{cases} \text{for all } k \in [1, q] \text{ and } i \in [1, p] \text{ we have } W^*(i) \neq \emptyset \\ \text{and } a(k, i) = \min\{S[w](k, i)\colon S \in W^*(i)\} \end{cases} \tag{2.4}$$

and

$$\begin{cases} \text{for all } k \in [1, q] \text{ and } i \in [1, p] \text{ we have } W(i) \neq \emptyset \\ \text{and } a(k, i) = \min\{S[w](k, i)\colon S \in W(i)\} \end{cases} \tag{2.5}$$

where (2.3) and (2.4) follow from the above characterizations of $\hat{\text{GLB}}$, whereas (2.5) follows from (2.4) by noting that, given any $S \in \text{suve}(w)$ and $j \in [1, \text{card}(S)]$, upon letting $\bar{S}$ to be the set of the first j elements of S (i.e. upon taking $\bar{S} \subset S$ such that $\text{card}(\bar{S}) = j$ and every element of $\bar{S}$ is less than every element of $S \setminus \bar{S}$), we get $\bar{S} \in W(j)$ and $\bar{S}[w](1, j) = S[w](1, j)$.

Given any $w \in \text{pre}(q)$ we introduce nonnegative integers $\text{inc}(w)$ and $\text{dec}(w)$ which we may call the *length of the longest increasing* (resp: *decreasing*) *subsequence of* w and which we define by saying that if $\text{len}(w) = 0$ then $\text{inc}(w) = \text{dec}(w) = 0$, and if $\text{len}(w) \neq 0$ then $\text{inc}(w)$ (resp: $\text{dec}(w)$) equals the largest positive integer L for which there exists a sequence of integers $1 \leqslant i(1) < i(2) < \cdots < i(L) \leqslant \text{len}(w)$ such that $w(k, i(j)) < w(k, i(j+1))$ (resp: $w(k, i(j)) > w(k, i(j+1))$) for all $k \in [1, q]$ and $j \in [1, L-1]$, and we note that clearly

$$\text{inc}(w) = \text{len}(\text{veg}(w)) \text{ and } \text{dec}(w) = \text{len}(\text{veg}(\text{op}(w))). \tag{2.6}$$

Given any a and a' in $\text{pre}(q)$, we define $a <^* a'$ to mean that $0 \neq \text{len}(a) \geqslant \text{len}(a')$ and $a(k, i) < a'(k, i)$ for all $k \in [1, q]$ and $i \in [1, \text{len}(a')]$, and we define $a \leqslant^* a'$ to mean that either $a = a'$ or $a <^* a'$ and we note that this is also a *partial order* on $\text{pre}(q)$.

Given any w and w^* in $\text{popre}(1)$, we say that w^* is *elementarily equivalent* to w if $\text{len}(w^*) = \text{len}(w)$ and there exists $n \in [0, \text{len}(w) - 3]$ such that either: $w(1, n+1) < w(1, n+3) \leqslant w(1, n+2)$ and

$$w^*(1, i) = \begin{cases} w(1, i) & \text{if } i \in [1, \text{len}(w)] \setminus [n+1, n+3] \\ w(1, n+2) & \text{if } i = n+1 \\ w(1, n+1) & \text{if } i = n+2 \\ w(1, n+3) & \text{if } i = n+3 \end{cases}$$

or: $w(1, n+3) \leqslant w(1, n+1) < w(1, n+2)$ and

$$w^*(1, i) = \begin{cases} w(1, i) & \text{if } i \in [1, \text{len}(w)] \backslash [n+1, n+3] \\ w(1, n+1) & \text{if } i = n+1 \\ w(1, n+3) & \text{if } i = n+2 \\ w(1, n+2) & \text{if } i = n+3. \end{cases}$$

Given any w and w^* in popre(1), we say that w^* is *equivalent* to w if there exists $h \in N$ and $w_i \in \text{popre}(1)$ for all $i \in [1, h+1]$ such that: $w_1 = w$, and $w_{h+1} = w^*$, and, for every $i \in [1, h]$, depending on i we have that either w_{i+1} is elementarily equivalent to w_i, or w_i is elementarily equivalent to w_{i+1}. Note that this gives an equivalence relation on popre(1).

Given any $d \in N$, by a *pretableau* T of *width* q and *depth* d we mean a sequence $T[e]_{1 \leqslant e \leqslant d}$ with $T[e] \in \text{pre}(q)$; we remark that $T[e]$ may be called the eth *row* of T. By a *pretableau* of *width* q we mean a pretableau T of width q and depth d for some $d \in N$, and we then put $\text{dep}(T) = d$. Given any pretableau T of width q, firstly we introduce the *length* of T which we denote by len(T) and which we define by putting

$$\text{len}(T) = \begin{cases} \max\{\text{len}(T[e]) : e \in [1, \text{dep}(T)]\} & \text{if dep}(T) \neq 0 \\ 0 & \text{if dep}(T) = 0 \end{cases}$$

and secondly we introduce the *area* of T which we denote by are(T) and which we define by putting

$$\text{are}(T) = \sum_{e \in [1, \text{dep}(T)]} \text{len}(T[e])$$

and thirdly for any $m \in Z(q)$ we define

$$T \leqslant m \text{ to mean } T[e] \leqslant m \text{ for all } e \in [1, \text{dep}(T)]$$

and we may express this by saying that T is *bounded* by m, and fourthly for any $a \in \text{pre}(q)$ we define

$$a \leqslant T \text{ to mean } a \leqslant T[e] \text{ for all } e \in [1, \text{dep}(T)]$$

and we may express this by saying that T is *predominated* by a, and fifthly for every $k \in [1, q]$ we define $k(T)$ to be the unique pretableau of width 1 such that $\text{dep}(k(T)) = \text{dep}(T)$ and $k(T)[e] = k(T[e])$ for all $e \in [1, \text{dep}(T)]$, and we note that $k(T)$ may be called the kth *side* of T.

By a *tableau* of *width* q we mean a pretableau T of width q such that $T[e] \in \text{vec}(q)$ for all $e \in [1, \text{dep}(T)]$. Given any tableau T of width q we say that T is *quasistandard* if $T[e] \leqslant T[e+1]$ for all $e \in [1, \text{dep}(T) - 1]$, and we say that T is *standard* if T is quasistandard and $\text{len}(T[e]) > 0$ for all $e \in [1, \text{dep}(T)]$. By pab(q) (resp: popab(q), tab(q), stab(q)) we denote the set of all pretableaux (resp: positive pretableaux, tableaux, standard tableaux) of width q, and, for every $V \in N$, by pib$[q, V]$ (resp: popib$[q, V]$, tib$[q, V]$, stib$[q, V]$) we denote the set of all T in pab(q) (resp: popab(q), tab(q), stab(q)) such that are$(T)=V$. Given any $m \in Z(q)$, by pab(q, m) (resp: popab(q, m), tab(q, m), stab(q, m)) we denote the set of all T in pab(q) (resp: popab(q), tab(q), stab(q)) such that $T \leqslant m$, and, for every $V \in N$, by pib(q, m, V) (resp: popib(q, m, V), tib(q, m, V), stib(q, m, V)) we denote the set of all T in pab(q, m) (resp: popab(q, m), tab(q, m), stab(q, m)) such that are$(T)=V$. By stab$_0(q)$ we denote the unique standard tableau of width q and depth 0.

Given any $m \in Z(q)$ and $p \in N$, firstly we put

$$\operatorname{stab}(q, m, p)=\{T \in \operatorname{stab}(q, m): \operatorname{len}(T) \leqslant p\}$$

and secondly for every $V \in N$ we put

$$\operatorname{stab}[q, m, p, V]=\{T \in \operatorname{stab}(q, m, p): \operatorname{are}(T)=V\}$$

and thirdly for every $a \in \operatorname{vec}(q, m, p)$ we put

$$\operatorname{stab}(q, m, p, a)=\{T \in \operatorname{stab}(q, m): a \leqslant T\}$$

and fourthly for every $a \in \operatorname{vec}(q, m, p)$ and $V \in N$ we put

$$\operatorname{stab}(q, m, p, a, V)=\{T \in \operatorname{stab}(q, m, p, a): \operatorname{are}(T)=V\}.$$

Given any $T \in$ stab(q) and $e \in N^*$, by $[T, e]$ we denote the unique member of vec(q) such that

$$\operatorname{len}([T, e])=\begin{cases}\operatorname{len}(T[e]) & \text{if } e \in[1, \operatorname{dep}(T)] \\ 0 & \text{if } e \in N^* \backslash[1, \operatorname{dep}(T)]\end{cases}$$

and

$$[T, e](k, i)=T[e](k, i) \text{ for all } k \in[1, q] \text{ and } i \in[1, \operatorname{len}([T, e])]$$

and by (T, e) we denote the unique member of covec(q) such that

$$\operatorname{len}((T, e))=\operatorname{card}(\{i \in[1, \operatorname{dep}(T)]: \operatorname{len}(T[i]) \geqslant e\})$$

and

$$(T, e)(k, i)=T[i](k, e) \text{ for all } k \in[1, q] \text{ and } i \in[1, \operatorname{len}((T, e))]$$

and we remark that (T, e) may be called the eth *column* of T, and we note that for all $e \in[1, \operatorname{len}(T)]$ we have $\operatorname{len}((T, e)) \neq 0$, and for all $e \in[1, \operatorname{len}(T)-1]$ we have $(T, e)<^*(T, e+1)$. Conversely, given any d in N and $T_1<^* T_2<^* \cdots <^* T_d$ in covec$(q) \backslash\{\operatorname{vec}_0(q)\}$, clearly there exists a unique T in stab(q) with len$(T)=d$ such that $(T, e)=T_e$ for all $e \in[1, d]$.

By a *preunitableau* (resp: *unitableau*) we mean a pretableau (resp: tableau) of width 1. By a *prebitableau* (resp: *bitableau*) we mean a pretableau (resp: tableau) of width 2.

Given any $T \in$ stab(1), by vas(T) we denote the unique member of popre$[1, \operatorname{are}(T)]$ such that for all $f \in[1, \operatorname{dep}(T)]$ and $j \in[1, \operatorname{len}(T[f])]$ we have

$$\operatorname{vas}(T)\Big(1, j+\sum_{e \in[f+1, \operatorname{dep}(T)]} \operatorname{len}(T[e])\Big)=T[f](1, j)$$

and we remark that vas(T) may be called the *vectorial associate* of T.

Again, given any $T \in \mathrm{stab}(1)$, by covas(T) we denote the unique member of popre$[1, \mathrm{are}(T)]$ such that for all $f \in [1, \mathrm{len}(T)]$ and $j \in [1, \mathrm{len}((T, f))]$ we have

$$\mathrm{covas}(T)\Big(1, 1 - j + \sum_{e \in [1,f]} \mathrm{len}((T, e))\Big) = (T, f)(1, j)$$

and we remark that covas(T) may be called the *covectorial associate* of T.

By rec(q) we denote the set of all pairs (k, j) with $k \in [1, q]$ and $j \in Z$; we call rec(q) the *infinite integral rectangle of width q*. By a *protovector of width q* we mean a mapping t: $\mathrm{rec}(q) \to N$ which, to each $k \in [1, q]$ and $j \in Z$, associates $t(k, j) \in N$, and by proc(q) we denote the set of all protovectors of width q. For any $a \in \mathrm{pre}(q)$, we introduce the *vectorial content* of a which we denote by con$[a]$ and which we define by saying that con$[a]$ is the unique member of proc(q) such that for all $k \in [1, q]$ and $j \in Z$ we have

$$\mathrm{con}[a](k, j) = \mathrm{card}(\{i \in [1, \mathrm{len}(a)]: a(k, i) = j\}).$$

For any $T \in \mathrm{pab}(q)$ we introduce the *content* of T which we denote by con(T) and which we define by saying that con(T) is the unique member of proc(q) such that for all $k \in [1, q]$ and $j \in Z$ we have

$$\mathrm{con}(T)(k, j) = \sum_{e \in [1, \mathrm{dep}(T)]} \mathrm{con}[T[e]](k, j).$$

Given any $m \in Z(q)$, by cub(q, m) we denote the set of all $y \in Z(q)$ such that $y(k) \in [1, m(k)]$ for all $k \in [1, q]$; we call cub(q, m) the *q-dimensional positive integral cube bounded by m*. Given any $m \in Z(q)$, by mon(q, m) we denote the set of all maps t: $\mathrm{cub}(q, m) \to N$, and we remark that members of mon(q, m) may be called *protomonomials* on cub(q, m) where the word protomonomial is meant to suggest the exponent system of a monomial; for example: if $(X_y)_{y \in \mathrm{cub}(q,m)}$ is a family of indeterminates then

$$\prod_{y \in \mathrm{cub}(q,m)} X_y^{t(y)}$$

is the monomial corresponding to t. Given any $m \in Z(q)$ and given any t and t' in mon(q, m), we define $t' \leq t$ to mean that, in the above notation, the monomial corresponding to t is divisible by the monomial corresponding to t', i.e. to mean that $t'(y) \leq t(y)$ for all $y \in \mathrm{cub}(q, m)$, and we remark that this makes mon(q, m) into a lattice.

Given any $m \in Z(q)$, firstly for every $t \in \mathrm{mon}(q, m)$ we define

$$\mathrm{supp}(t) = \{y \in \mathrm{cub}(q, m): t(y) \neq 0\}$$

and

$$\mathrm{abs}(t) = \sum_{y \in \mathrm{cub}(q,m)} t(y)$$

where supp and abs are meant to suggest *support* and *absolute value* respectively, and secondly for every $V \in N$ we put

$$\mathrm{mon}[[q, m, V]] = \{t \in \mathrm{mon}(q, m) \colon \mathrm{abs}(t) = V\}$$

and thirdly for every $w \in \mathrm{popre}(q, m)$ we define $\mathrm{mos}[w, m]$ to be the unique element in $\mathrm{mon}(q, m)$ such that in the above notation we have

$$\prod_{i \in [1, \mathrm{len}(w)]} X_{w[i]} = \prod_{y \in \mathrm{cub}(q,m)} x_y^{\mathrm{mos}[w,m](y)}$$

i.e. such that for all $y \in \mathrm{cub}(q, m)$ we have

$$\mathrm{mos}[w, m](y) = \mathrm{card}(\{i \in [1, \mathrm{len}(w)] \colon w[i] = y\})$$

and we remark that $\mathrm{mos}[w, m]$ may be called the *monomial associate* of (w, m).

For any finite subset Y of $Z(q)$, by $\mathrm{ind}(Y)$ we denote the largest nonnegative integer j for which there exist elements $y_1, \ldots, y_j$ in Y such that $y_i(k) < y_{i+1}(k)$ for all $i \in [1, j-1]$ and $k \in [1, q]$; we call $\mathrm{ind}(Y)$ the *index* of Y.

Given any $m \in Z(q)$ and $p \in N$, firstly we put

$$\mathrm{mon}(q, m, p) = \{t \in \mathrm{mon}(q, m) \colon \mathrm{ind}(\mathrm{supp}(t)) \leq p\}$$

and secondly for every $V \in N$ we put

$$\mathrm{mon}[q, m, p, V] = \{t \in \mathrm{mon}(q, m, p) \colon \mathrm{abs}(T) = V\}$$

and thirdly for every $a \in \mathrm{vec}(q, m, p)$ we put

$$\mathrm{mon}(q, m, p, a) = \{t \in \mathrm{mon}(q, m, p) \colon \text{for all } k \in [1, q] \text{ and } i \in [1, \mathrm{len}(a)] \text{ we have } \mathrm{ind}(\{y \in \mathrm{supp}(t) \colon y(k) < a(k, i)\}) < i\}$$

and fourthly for every $a \in \mathrm{vec}(q, m, p)$ and $V \in N$ we put

$$\mathrm{mon}(q, m, p, a, V) = \{t \in \mathrm{mon}(q, m, p, a) \colon \mathrm{abs}(t) = V\}.$$

Given any $m \in Z(q)$ and $t \in \mathrm{mon}(q, m)$, to characterize $\mathrm{ind}(\mathrm{supp}(t))$ and to find the largest a in $\mathrm{vec}(q, m)$ such that $t \in \mathrm{mon}(q, m, \mathrm{len}(a), a)$, we define $\mathrm{veg}(t) \in \mathrm{vec}(q, m)$ by putting

$$\mathrm{veg}(t) = \mathrm{GLB}(\{w \in \mathrm{vec}(q, m) \colon \mathrm{mos}[w, m] \leq t\})$$

and we remark that $\mathrm{veg}(t)$ may be called the *vectorial* GLB of t and we note that

$$\mathrm{ind}(\mathrm{supp}(t)) = \mathrm{len}(\mathrm{veg}(t)) \tag{2.7}$$

and for any $p \in N$ and $a \in \mathrm{vec}(q, m, p)$ we have

$$t \in \mathrm{mon}(q, m, p, a) \Leftrightarrow a \leq \mathrm{veg}(t). \tag{2.8}$$

3. Roinsertion in univectors

Definition 3.1. Given any $z \in N^*$ and $a \in \mathrm{vec}(1)$, upon letting $p = \mathrm{len}(a)$, first pictorially and then more precisely, we shall define certain elements $\mathrm{RE}(z, a) \in N^* \cup \{\infty\}$, $\mathrm{RP}(z, a) \in [1, p+1]$, and $R(z, a) \in \mathrm{vec}(1)$, which we may respectively call the *roinsertive entry* of (z, a), the *roinsertive place* of (z, a), and the *roinsertion* of (z, a); here *ro* is meant to suggest row. Pictorially we define

$$\mathrm{RE}(z, a) = x, \quad \mathrm{RP}(z, a) = t \quad \text{and} \quad R(z, a) = b$$

where

$$\begin{array}{ccccccccc} a(1,1) < \cdots < a(1,t-1) < z \leqslant x = a(1,t) < a(1,t+1) < \cdots < & a(1,p) \\ \| \qquad\qquad \| \qquad\qquad \| \qquad\qquad\qquad \| \qquad\qquad & \| \\ b(1,1) < \cdots < b(1,t-1) < b(1,t) \qquad\qquad < b(1,t+1) < \cdots < & b(1,p) \end{array}$$

or

$$\begin{array}{l} a(1,1) < \cdots < a(1,p) < z \qquad < x = \infty \\ \| \qquad\qquad \| \qquad\quad \| \\ b(1,1) < \cdots < b(1,p) < b(1, t = p+1). \end{array}$$

More precisely, firstly if $z \leqslant a(1, i)$ for some $i \in [1, p]$ then we note that clearly there exists a unique $\bar{x} \in N^*$ and $\bar{t} \in [1, p]$ such that

$$\begin{cases} z \leqslant \bar{x} = a(1, \bar{t}) \\ \text{and } a(1, i) < z \text{ for all } i \in [1, \bar{t} - 1] \\ \text{and } \bar{x} < a(1, j) \text{ for all } j \in [\bar{t} + 1, p] \end{cases}$$

and secondly (without assuming any condition on z) we note that clearly there exists a unique $x \in N^* \cup \{\infty\}$, $t \in [1, p+1]$, and $b \in \mathrm{vec}(1)$ such that

$$t = \begin{cases} \bar{t} & \text{if } z \leqslant a(1, i) \text{ for some } i \in [1, p] \\ p + 1 & \text{if } a(1, i) < z \text{ for all } i \in [1, p] \end{cases}$$

and

$$x = \begin{cases} \bar{x} & \text{if } z \leqslant a(1, i) \text{ for some } i \in [1, p] \\ \infty & \text{if } a(1, i) < z \text{ for all } i \in [1, p] \end{cases}$$

and

$$\mathrm{len}(b) = \begin{cases} p & \text{if } z \leqslant a(1, i) \text{ for some } i \in [1, p] \\ p + 1 & \text{if } a(1, i) < z \text{ for all } i \in [1, p] \end{cases}$$

and

$$b(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, \mathrm{len}(b)] \setminus \{t\} \\ z & \text{if } i = t \in [1, \mathrm{len}(b)] \end{cases}$$

and we define $\mathrm{RE}(z, a) \in N^* \cup \{\infty\}$, $\mathrm{RP}(z, a) \in [1, p+1]$, and $R(z, a) \in \mathrm{vec}(1)$ by putting

$$\mathrm{RE}(z, a) = x, \quad \mathrm{RP}(z, a) = t, \quad \text{and} \quad R(z, a) = b.$$

Definition 3.2. Given any $w \in \mathrm{popre}(1)$ and $a \in \mathrm{vec}(1)$, upon letting $r = \mathrm{len}(w)$, firstly we note that there exists a unique sequence $A = A(i)_{1 \leqslant i \leqslant r+1}$, with $A(1), \ldots, A(r+1)$, in $\mathrm{vec}(1)$, such that $A(1) = a$ and $A(i+1) = R(w(1, i), A(i))$ for all $i \in [1, r]$, and there exists a unique sequence $X = X(i)_{1 \leqslant i \leqslant r}$, with $X(1), \ldots, X(r)$ in $N^* \cup \{\infty\}$, such that $X(i) = \mathrm{RE}(w(1, i), A(i))$ for all $i \in [1, r]$, and we define $\mathrm{RQ}(w, a)$ and $\mathrm{RE}(w, a)$ by putting $\mathrm{RQ}(w, a) = A$ and $\mathrm{RE}(w, a) = X$, and we remark that $\mathrm{RQ}(w, a)$ and $\mathrm{RE}(w, a)$ may respectively be called the *sequential roinsertion* of (w, a) and the *roinsertive entry* of (w, a), and secondly we note that there exists a unique $\bar{X} \in \mathrm{popre}(1)$ such that $\mathrm{len}(\bar{X}) = \mathrm{card}(\{i \in [1, r]: X(i) \neq \infty\})$ and such that upon considering the unique increasing bijection $[1, \mathrm{len}(\bar{X})] \to \{i \in [1, r]: X(i) \neq \infty\}$ we have that $\bar{X}(1, i) = X$ (image of i under the said bijection) for all $i \in [1, \mathrm{len}(\bar{X})]$, and we define $\mathrm{REF}(w, a) \in \mathrm{popre}(1)$ by putting $\mathrm{REF}(w, a) = \bar{X}$ and we remark that $\mathrm{REF}(w, a)$ may be called the *finite roinsertive entry* of (w, a), and thirdly we define $R(w, a) \in \mathrm{vec}(1)$ by putting $R(w, a) = A(r + 1)$ and we remark that $R(w, a)$ may be called the *roinsertion* of (w, a).

Again given any $w \in \mathrm{popre}(1)$, we define $\mathrm{RL}(w) \in \mathrm{vec}(1)$ by putting $\mathrm{RL}(w) = R(w, \mathrm{vec}_0(1))$ and we remark that $\mathrm{RL}(w)$ may be called the *vectorial roinsertion* of w.

Lemma 3.3. *In the situation of* (3.1), *assume that $z \leqslant a(1, i)$ for some $i \in [1, p]$, and let $a' \in \mathrm{vec}(1)$ be such that $a \leqslant a'$. Then we have the following.*

(3.3.1) *Upon letting $t' = \mathrm{RP}(x, a')$ we have $t' \leqslant t$.*

(3.3.2) *Upon letting $b' = R(x, a')$ we have $b \leqslant b'$.*

Proof. Since $z \leqslant a(1, i)$ for some $i \in [1, p]$, we must have $t \in [1, p]$ and $x = a(1, t)$ and $\mathrm{len}(b) = p$. Since $a \leqslant a'$, upon letting $p' = \mathrm{len}(a')$, we get $p' \leqslant p$ and $a(1, i) \leqslant a'(1, i)$ for all $i \in [1, p']$. By the definition of t', for every $i \in [1, t' - 1]$ we have $i \in [1, p']$ and $a'(1, i) < x$; therefore we must have $t' \leqslant t$, and hence we get

$$\mathrm{len}(b') = \begin{cases} p' \leqslant p & \text{if } t' \in [1, p'] \\ p' + 1 = t' \leqslant t \leqslant p & \text{if } t' \notin [1, p']. \end{cases}$$

Now

$$b(1, i) = \begin{cases} a(1, i) \leqslant a'(1, i) = b'(1, i) & \text{if } i \in [1, \mathrm{len}(b')] \setminus \{t', t\} \\ a(1, i) < x = b'(1, i) & \text{if } t > t' = i \in [1, \mathrm{len}(b')] \\ z \leqslant x = b'(1, i) & \text{if } t' = t = i \in [1, \mathrm{len}(b')] \\ z \leqslant x \leqslant a'(1, i) = b'(1, i) & \text{if } t' < t = i \in [1, \mathrm{len}(b')] \end{cases}$$

and hence $b \leqslant b'$. $\square$

Lemma 3.4. *In the situation of* (3.1), *given any $z^* \in N^*$, upon letting $t^* = \mathrm{RP}(z^*, b)$ and $x^* = \mathrm{RE}(z^*, b)$, we have the following.*

(3.4.1) *If $z^* \leqslant z$ then $t^* \leqslant t$ and $\infty \neq x^* \leqslant x$.*

(3.4.2) *If $z < z^*$ then $t < t^*$.*

(3.4.3) *If $z < z^*$ and $x^* \neq \infty$ then $x < x^*$.*

Proof. By the definition of t, x and b we see that

(1) $t \in [1, \mathrm{len}(b)]$ and $b(1, t) = z \leq x$

and

(2) $x < b(1, \bar{t})$ for all $\bar{t} \in [t+1, \mathrm{len}(b)]$.

If $z^* \leq z$ then, in view of (1), by the definition of t^* and x^* we get $t^* \in [1, \mathrm{len}(b)]$ and $t^* \leq t$ and

$$z^* \leq x^* = b(1, t^*) \leq b(1, t) = z \leq x$$

and hence $\infty \neq x^* \leq x$.

If $z < z^*$ then, in view of (1), by the definition of t^* we see that $t < t^*$, and, if also $x^* \neq \infty$, then by the definition of x^* we see that

$$t^* \in [1, \mathrm{len}(b)] \quad \text{and} \quad z^* \leq x^* = b(1, t^*)$$

and hence, in view of (2), we get $x < x^*$. □

Lemma 3.5. *In the situation of* (3.1), *assume that* $a = \mathrm{veg}(w)$ *for some* $w \in \mathrm{popre}(1)$. *Let* $r = \mathrm{len}(w)$, *and let* $y \in \mathrm{popre}[1, r+1]$ *be such that*

$$y(1, i) = \begin{cases} w(1, i) & \text{if } i \in [1, r] \\ z & \text{if } i = r+1. \end{cases}$$

Then $b = \mathrm{veg}(y)$.

Proof. For every $i \in N$ let

$$W(i) = \{S \in \mathrm{suve}(w) : \mathrm{card}(S) = i\}$$

$$\hat{W}(i) = \{S \in W(i) : w(1, s) < z \text{ for all } s \in S\}$$

$$Y(i) = \{S \in \mathrm{suve}(y) : \mathrm{card}(S) = i\}$$

and note that, given any $S \in \hat{W}(i)$, upon letting $S^* = S \cup \{r+1\}$, we have $S^* \in Y(i+1)$ and

$$S^*[y](1, j) = \begin{cases} S[w](1, j) & \text{if } j \in [1, i] \\ z & \text{if } j = i+1. \end{cases}$$

Now clearly

(1) $W(0) = \{\emptyset\} = Y(0)$

and

(2) $Y(i+1) = W(i+1) \cup \{S^* : S \in \hat{W}(i)\}$ for all $i \in N$.

and by (2.3) and (2.5) we have

(3) $W(i) = \emptyset$ for all $i \in N^* \setminus [1, p]$

and

(4) $\begin{cases} W(i) \neq \emptyset \text{ and } a(1, i) = \min\{S[w](1, i) : S \in W(i)\} \\ \text{for all } i \in [1, p]. \end{cases}$

Upon letting $c = \mathrm{veg}(y)$, i.e. upon letting $c = \mathrm{GLB}(\{S[y] : S \in \mathrm{suve}(y)\})$, we see that

$$\begin{aligned} \operatorname{len}(c) \neq p &\Leftrightarrow \operatorname{len}(c) = p + 1 \text{ and } c(1, p + 1) = z \\ &\Leftrightarrow \hat{W}(p) \neq \emptyset \\ &\Leftrightarrow a(1, i) < z \text{ for all } i \in [1, p] \end{aligned}$$

where the first two "$\Leftrightarrow$" follow from (1) to (4), whereas the last "$\Rightarrow$" follows by taking any $S \in \hat{W}(p)$ and noting that then for all $i \in [1, p]$ we have $a(1, i) \leqslant S[w](1, i) < z$, and finally, in view of (4), the last "$\Leftarrow$" follows by noting that, in case of $p \neq 0$, we can find $S \in W(p)$ such that $a(1, p) = S[w](1, p)$ and then we would have $S[w](1, p) = a(1, p) < z$ and hence we would get $S \in \hat{W}(p)$. By the definition of b we also have

$$\begin{aligned} \operatorname{len}(b) \neq p &\Leftrightarrow \operatorname{len}(b) = p + 1 \text{ and } b(1, p + 1) = z \\ &\Leftrightarrow a(1, i) < z \text{ for all } i \in [1, p] \end{aligned}$$

and hence we get

$$(5) \quad \begin{cases} \operatorname{len}(b) \neq p \Leftrightarrow \operatorname{len}(b) = p + 1 \text{ and } b(1, p + 1) = z \\ \qquad\qquad \Leftrightarrow \operatorname{len}(c) = p + 1 \text{ and } c(1, p + 1) = z \\ \qquad\qquad \Leftrightarrow \operatorname{len}(c) \neq p. \end{cases}$$

By taking y for w in (2.5) we get

$$\begin{cases} Y(i) \neq \emptyset \text{ and } c(1, i) = \min\{S[y](1, i) \colon S \in Y(i)\} \\ \text{for all } i \in [1, \operatorname{len}(c)] \end{cases}$$

and hence by (1), (2) and (4) we get

$$(6) \quad c(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \text{ and either } a(1, i) < z \text{ or } \hat{W}(i - 1) = \emptyset \\ z & \text{if } i \in [1, p] \text{ and } z \leqslant a(1, i) \text{ and } \hat{W}(i - 1) \neq \emptyset. \end{cases}$$

By the definition of t we set that

(7) $a(1, i) < z$ for all $i \in [1, p]$ with $i < t$.

If $i \in [2, p]$ is such that $\hat{W}(i - 1) \neq \emptyset$ then upon taking $S \in \hat{W}(i - 1)$ we get $a(1, i - 1) \leqslant S[w](1, i - 1) < z$ and hence by the definition of t we get $i \leqslant t$; therefore

(8) $\hat{W}(i - 1) = \emptyset$ for all $i \in [t + 1, p]$.

By the definition of t we see that if $t \in [2, p]$ then $a(1, t - 1) < z$ and by (4) we can take $S \in W(t - 1)$ with $S[w](1, t - 1) = a(1, t - 1)$ and now we get $S[w](1, t - 1) < z$ and hence $S \in \hat{W}(t - 1)$ and therefore $\hat{W}(t - 1) \neq \emptyset$; by the definition of t we also know that if $t \in [1, p]$ then $z < a(1, t)$; consequently, in view of (1) we conclude that

(9) if $t \in [1, p]$ then $z \leqslant a(1, t)$ and $\hat{W}(t - 1) \neq \emptyset$.

In view of (7), (8), (9), by the definition of b we see that

$$b(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \text{ and either } a(1, i) < z \text{ or } \hat{W}(i - 1) = \emptyset \\ z & \text{if } i \in [1, p] \text{ and } z \leqslant a(1, i) \text{ and } \hat{W}(i - 1) \neq \emptyset \end{cases}$$

and therefore by (5) and (6) we get $c = b$. $\square$

Theorem 3.6. *For every* $w \in \text{popre}(1)$ *we have* $\text{RL}(w) = \text{veg}(w)$, *and hence in particular we have* $\text{len}(\text{RL}(w)) = \text{inc}(w)$.

Proof. If $\text{len}(w) = 0$, i.e. if $w = \text{vec}_0(1)$, then obviously $\text{RL}(w) = w = \text{veg}(w)$. Therefore, in view of (3.5), by induction on $\text{len}(w)$ we get $\text{RL}(w) = \text{veg}(w)$ and hence, in view of (2.6) we get $\text{len}(\text{RL}(w)) = \text{inc}(w)$. $\square$

Lemma 3.7. *In the situation of* (3.1), *let* $e \in \text{popre}[1, p+1]$ *be such that* $e(1, i) = a(1, i)$ *for all* $i \in [1, p]$, *and* $e(1, p+1) = z$. *Assume that* $x \neq \infty$, *and let* $e^* \in \text{popre}[1, p+1]$ *be such that* $e^*(1, 1) = x$, *and* $e^*(1, i+1) = b(1, i)$ *for all* $i \in [1, p]$. *Then* e^* *is equivalent to* e.

Proof. For every $n \in [-1, t-2]$ let $b_n \in \text{popre}[1, p+1]$ be defined by putting

$$b_n(1, i) = \begin{cases} b(1, i) & \text{if } i \in [1, n+1] \\ x & \text{if } i = n+2 \\ b(1, i-1) & \text{if } i \in [n+3, p+1] \end{cases}$$

and for every $n \in [t-2, p-2]$ let $a_n \in \text{popre}[1, p+1]$ be defined by putting

$$a_n(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, n+2] \\ z & \text{if } i = n+3 \\ a(1, i-1) & \text{if } i \in [n+4, p+1]. \end{cases}$$

Now firstly for every $n \in [0, t-2]$ we have $b_n(1, n+1) < b_n(1, n+3) \leqslant b_n(1, n+2)$ and

$$b_{n-1}(1, i) = \begin{cases} b_n(1, i) & \text{if } i \in [1, p+1] \backslash [n+1, n+3] \\ b_n(1, n+2) & \text{if } i = n+1 \\ b_n(1, n+1) & \text{if } i = n+2 \\ b_n(1, n+3) & \text{if } i = n+3 \end{cases}$$

and hence b_{n-1} is elementarily equivalent to b_n; therefore b_{-1} is equivalent to b_{t-2}. Secondly for every $n \in [t-1, p-2]$ we have $a_n(1, n+3) \leqslant a_n(1, n+1) < a_n(1, n+2)$ and

$$a_{n-1}(1, i) = \begin{cases} a_n(1, i) & \text{if } i \in [1, p+1] \backslash [n+1, n+3] \\ a_n(1, n+1) & \text{if } i = n+1 \\ a_n(1, n+3) & \text{if } i = n+2 \\ a_n(1, n+2) & \text{if } i = n+3 \end{cases}$$

and hence a_{n-1} is elementarily equivalent to a_n; therefore a_{t-2} is equivalent to

a_{p-2}. Thirdly we obviously have

$$e^* = b_{-1} \quad \text{and} \quad b_{t-2} = a_{t-2} \quad \text{and} \quad a_{p-2} = e$$

and therefore e^* is equivalent to e. $\square$

Lemma 3.8. *Given any $\tilde{a}$ in* vec(1), *and $\tilde{w}$ and $\tilde{w}^*$ in* popre(1), *we have that: if $\tilde{w}^*$ is equivalent to $\tilde{w}$ then $R(\tilde{w}^*, \tilde{a}) = R(\tilde{w}, \tilde{a})$ and* $\mathrm{REF}(\tilde{w}^*, \tilde{a})$ *is equivalent to* $\mathrm{REF}(\tilde{w}, \tilde{a})$.

Proof. By induction on the number of elementary equivalences, or their inverses, required for converting $\tilde{w}$ into $\tilde{w}^*$, our assertion follows from the *claim* which says that if $\tilde{w}^*$ is elementarily equivalent to $\tilde{w}$ then $R(\tilde{w}^*, \tilde{a}) = R(\tilde{w}, \tilde{a})$ and either $\mathrm{REF}(\tilde{w}^*, \tilde{a}) = \mathrm{REF}(\tilde{w}, \tilde{a})$ or $\mathrm{REF}(\tilde{w}^*, \tilde{a})$ is elementarily equivalent to $\mathrm{REF}(\tilde{w}, \tilde{a})$. In turn, this claim follows from the somewhat *stronger claim* which says that, given any $n \in [0, r-3]$, where $r = \mathrm{len}(\tilde{w})$, upon letting

$$a = R(\tilde{w}(1, n), R(\tilde{w}(1, n-1), \ldots, R(\tilde{w}(1, 2), R(\tilde{w}(1, 1), \tilde{a}))\cdots))$$

where it is understood that $a = \tilde{a}$ if $n = 0$, and upon letting w, w' and w'' to be the members of popre[1, 3] obtained by putting

$$w(1, i) = \tilde{w}(1, n+i) \text{ for all } i \in [1, 3]$$

and

$$w'(1, i) = \begin{cases} w(1, 2) & \text{if } i = 1 \\ w(1, 1) & \text{if } i = 2 \\ w(1, 3) & \text{if } i = 3 \end{cases} \quad \text{and} \quad w''(1, i) = \begin{cases} w(1, 1) & \text{if } i = 1 \\ w(1, 3) & \text{if } i = 2 \\ w(1, 2) & \text{if } i = 3, \end{cases}$$

and upon letting $X = X(i)_{1 \leq i \leq 3}$, $X' = X'(i)_{1 \leq i \leq 3}$ and $X'' = X''(i)_{1 \leq i \leq 3}$ to be the sequences with entries in $N^* \cup \{\infty\}$ obtained by putting

$$X = \mathrm{RE}(w, a)$$

and

$$X'(i) = \begin{cases} X(2) & \text{if } i = 1 \\ X(1) & \text{if } i = 2 \\ X(3) & \text{if } i = 3 \end{cases} \quad \text{and} \quad X''(i) = \begin{cases} X(1) & \text{if } i = 1 \\ X(3) & \text{if } i = 2 \\ X(2) & \text{if } i = 3, \end{cases}$$

we have the following:

(′) If $w(1, 1) < w(1, 3) \leq w(1, 2)$ then $R(w', a) = R(w, a)$ and either

(1′) $X(1) < X(3) \leq X(2) < \infty$ and $\mathrm{REF}(w', a) = X'$

or

(2′) $X(3) \leq X(1) < X(2) < \infty$ and $\mathrm{REF}(w', a) = X''$

or

(3′) $\mathrm{len}(\mathrm{REF}(w, a)) \in [1, 2]$ and $\mathrm{REF}(w', a) = \mathrm{REF}(w, a)$.

(″) If $w(1, 3) \leq w(1, 1) < w(1, 2)$ then $R(w'', a) = R(w, a)$ and either

(1″) $X(3) \leq X(1) < X(2) < \infty$ and $\mathrm{REF}(w'', a) = X''$

or

(2″) $\mathrm{len}(\mathrm{REF}(w, a)) \in [1, 2]$ and $\mathrm{REF}(w'', a) = \mathrm{REF}(w, a)$.

To prove the stronger claim, let $z = w(1, 1)$ and let the rest of the notation be as in (3.1). Also let

$$z^* = w(1, 2), \quad t^* = \mathrm{RP}(z^*, a), \quad z^{**} = w(1, 3)$$

and

$$b^* = R(z^*, b), \quad b^{**} = R(z^{**}, b^*), \quad x^* = \mathrm{RE}(z^*, b), \quad x^{**} = \mathrm{RE}(z^{**}, b^*)$$

and

$$\begin{cases} b'^* = R(z^*, a), & b' = R(z, b'^*), \quad b'^{**} = R(z^{**}, b'), \\ x'^* = \mathrm{RE}(z^*, a), & x' = \mathrm{RE}(z, b'^*), \quad x'^{**} = \mathrm{RE}(z^{**}, b') \end{cases}$$

and

$$\begin{cases} b'' = R(z, a) = b, & b''^{**} = R(z^{**}, b''), \quad b''^{*} = R(z^*, b''^{**}), \\ x'' = \mathrm{RE}(z, a) = x, & x''^{**} = \mathrm{RE}(z^{**}, b''), \quad x''^{*} = \mathrm{RE}(z^*, b''^{**}). \end{cases}$$

We shall now divide the argument into five cases according to: firstly when $t < t^* \leq p$; secondly when $t \leq p < t^*$; thirdly when $z < z^*$ and $p < t = t^*$; fourthly when $z < z^*$ and $t = t^* = p$; and fifthly when $z < z^*$ and $t = t^* < p$.

Now firstly if $t < t^* \leq p$ then

$$\begin{cases} a(1, 1) < \cdots < a(1, t-1) < z \leq x = a(1, t) < \cdots < a(1, t^*-1) < z^* \\ \text{and } z^* \leq x'^* = a(1, t^*) \end{cases}$$

(where the expression $a(1, 1) < \cdots < a(1, t-1)$ is understood to be vacuous in case $t = 1$) and $\mathrm{len}(b) = p$ and

$$b(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \setminus \{t\} \\ z & \text{if } i = t \end{cases}$$

and $\mathrm{len}(b'^*) = p$ and

$$b'^*(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \setminus \{t^*\} \\ z^* & \text{if } i = t^* \end{cases}$$

and hence

$$\mathrm{RP}(\tilde{z}, b) \leq t < t^* = \mathrm{RP}(z^*, b) \leq p \text{ for all } \tilde{z} \in [1, z]$$

and $\mathrm{len}(b^*) = p$ and $x^* = x'^*$ and

$$b^*(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \setminus \{t, t^*\} \\ z & \text{if } i = t \\ z^* & \text{if } i = t^* \end{cases}$$

and

$$b^*(1, t) = z \leq x < b^*(1, t+1) \leq z^* = b^*(1, t^*) \leq x^* < \infty$$

and $\mathrm{RP}(z, b'^*) = t$ and $\mathrm{len}(b') = p$ and $x' = x$ and

$$b'(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \setminus \{t, t^*\} \\ z & \text{if } i = t \\ z^* & \text{if } i = t^*. \end{cases}$$

Thus

$$(1) \quad \begin{cases} \text{if } t < t^* \leqslant p \text{ then } \mathrm{len}(b) = p \\ \text{and } \mathrm{RP}(\tilde{z}, b) \leqslant t < t^* = \mathrm{RP}(z^*, b) \leqslant p \text{ for all } \tilde{z} \in [1, z] \end{cases}$$

and

$$(2) \quad \begin{cases} \text{if } t < t^* \leqslant p \text{ then } \mathrm{len}(b^*) = p \\ \text{and } b^*(1, t) = z \leqslant x < b^*(1, t+1) \leqslant z^* = b^*(1, t^*) \leqslant x^* < \infty \end{cases}$$

and

(3) if $t < t^* \leqslant p$ then $b' = b^*$ and $x' = x$ and $x'^* = x^*$.

Now on the one hand, if $z < z^{**} \leqslant z^*$ and $t < t^* \leqslant p$ then by (2) we get $x < x^{**} \leqslant x^* < \infty$; therefore in view of (3) we conclude that

$$(4) \quad \begin{cases} \text{if } z < z^{**} \leqslant z^* \text{ and } t < t^* \leqslant p \text{ then } b'^{**} = b^{**} \\ \text{and } x < x^{**} \leqslant x^* < \infty \text{ and } (x'^*, x', x'^{**}) = (x^*, x, x^{**}) \end{cases}$$

where the last equality is meant as an equality of sequences. On the other hand, if $z^{**} \leqslant z < z^*$ and $t < t^* \leqslant p$ then by (2) we get $x^{**} \leqslant x < x^* < \infty$, and by (1) we see that $\mathrm{RP}(z^{**}, b) < \mathrm{RP}(z^*, b) \leqslant \mathrm{len}(b)$ and hence upon taking (b, z^*, z^{**}) for (a, z, z^*) in (3) we get $b''^* = b^{**}$ and $x''^* = x^*$ and $x''^{**} = x^{**}$; therefore

$$(5) \quad \begin{cases} \text{if } z^{**} \leqslant z < z^* \text{ and } t < t^* \leqslant p \text{ then } b''^* = b^{**} \\ \text{and } x^{**} \leqslant x < x^* < \infty \text{ and } (x'', x''^{**}, x''^*) = (x, x^{**}, x^*). \end{cases}$$

Secondly if $t \leqslant p < t^*$ then

$$a(1, 1) < \cdots < a(1, t-1) < z \leqslant x = a(1, t) < \cdots < a(1, p) < z^*$$

and $\mathrm{len}(b) = p$ and $x \neq \infty$ and

$$b(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \setminus \{t\} \\ z & \text{if } i = t \end{cases}$$

and $\mathrm{len}(b'^*) = p + 1 = t^*$ and $x'^* = \infty$ and

$$b'^*(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p+1] \setminus \{t^*\} \\ z^* & \text{if } i = t^* \end{cases}$$

and hence

$$\mathrm{RP}(\tilde{z}, b) \leqslant t \leqslant p < t^* = \mathrm{RP}(z^*, b) \text{ for all } \tilde{z} \in [1, z]$$

and $\mathrm{len}(b^*) = p + 1$ and $x^* = \infty$ and

$$b^*(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p+1] \setminus \{t, t^*\} \\ z & \text{if } i = t \\ z^* & \text{if } i = t^* \end{cases}$$

and

$$b^*(1, t) = z \leqslant x < b^*(1, t+1) \leqslant z^* = b^*(1, t^*) < x^* = \infty$$

and $\mathrm{RP}(z, b'^*) = t$ and $\mathrm{len}(b') = p + 1$ and $x' = x$ and

$$b'(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p+1] \setminus \{t, t^*\} \\ z & \text{if } i = t \\ z^* & \text{if } i = t^*. \end{cases}$$

Thus

$$\text{(6)} \quad \begin{cases} \text{if } t \leqslant p < t^* \text{ then } \mathrm{len}(b) = p \\ \text{and } \mathrm{RP}(\tilde{z}, b) \leqslant t \leqslant p < t^* = \mathrm{RP}(z^*, b) \text{ for all } \tilde{z} \in [1, z] \end{cases}$$

and

$$\text{(7)} \quad \begin{cases} \text{if } t \leqslant p < t^* \text{ then } \mathrm{len}(b^*) = p + 1 = t^* \\ \text{and } b^*(1, t) = z \leqslant x < b^*(1, t+1) \leqslant z^* = b^*(1, t^*) < x^* = \infty \end{cases}$$

and

(8) if $t \leqslant p < t^*$ then $b' = b^*$ and $x' = x \neq \infty$ and $x'^* = x^* = \infty$.

Now on the one hand, if $z < z^{**} \leqslant z^*$ and $t \leqslant p < t^*$ then by (7) we get $x < x^{**} < x^* = \infty$; therefore in view of (8) we conclude that

$$\text{(9)} \quad \begin{cases} \text{if } z < z^{**} \leqslant z^* \text{ and } t \leqslant p < t^* \text{ then } b'^{**} = b^{**} \\ \text{and } x'^* = \infty = x^* \text{ and } x \neq \infty \neq x^{**} \text{ and } (x', x'^{**}) = (x, x^{**}). \end{cases}$$

On the other hand, if $z^{**} \leqslant z < z^*$ and $t \leqslant p < t^*$ then by (7) we get $x^{**} \leqslant x < x^* = \infty$, and by (6) we see that $\mathrm{RP}(z^{**}, b) \leqslant \mathrm{len}(b) < \mathrm{RP}(z^*, b)$ and hence upon taking (b, z^*, z^{**}) for (a, z, z^*) in (8) we get $b''^* = b^{**}$ and $x''^* = x^*$ and $x''^{**} = x^{**}$; therefore

$$\text{(10)} \quad \begin{cases} \text{if } z^{**} \leqslant z < z^* \text{ and } t \leqslant p < t^* \text{ then } b''^* = b^{**} \\ \text{and } x''^* = \infty = x^* \text{ and } x \neq \infty \neq x^{**} \text{ and } (x'', x'^{**}) = (x, x^{**}). \end{cases}$$

Thirdly if $z < z^*$ and $p < t = t^*$ then

$$a(1, 1) < \cdots < a(1, p) < z < z^*$$

and $\mathrm{len}(b) = p + 1 = t$ and $x = \infty$ and

$$b(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \\ z & \text{if } i = p + 1 \end{cases}$$

and $\mathrm{len}(b'^*) = p + 1 = t^*$ and $x'^* = \infty$ and

$$b'^*(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \\ z^* & \text{if } i = p + 1 \end{cases}$$

and hence

$$\mathrm{RP}(\tilde{z}, b) \leqslant p + 1 < p + 2 = \mathrm{RP}(z^*, b) \text{ for all } \tilde{z} \in [1, z]$$

and $\mathrm{len}(b^*) = p + 2$ and $x^* = \infty$ and

$$b^*(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \\ z & \text{if } i = p + 1 \\ z^* & \text{if } i = p + 2 \end{cases}$$

and

$$b^*(1, p+1) = z < z^* = b^*(1, p+2) < x = x^* = \infty$$

and $\mathrm{RP}(z, b'^*) = t = p + 1 = \mathrm{len}(b')$ and $x' = z^*$ and

$$b'(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \\ z & \text{if } i = p + 1 \end{cases}$$

and therefore if also $z < z^{**} \leqslant z^*$ then: $\mathrm{len}(b^{**}) = p + 2$ and $x^{**} = z^*$ and

$$b^{**}(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \\ z & \text{if } i = p + 1 \\ z^{**} & \text{if } i = p + 2 \end{cases}$$

and $\mathrm{len}(b'^{**}) = p + 2$ and $x'^{**} = \infty$ and

$$b'^{**}(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \\ z & \text{if } i = p + 1 \\ z^{**} & \text{if } i = p + 2 \end{cases}$$

and hence $b'^{**} = b^{**}$. Thus

$$(11) \quad \begin{cases} \text{if } z < z^* \text{ and } p < t = t^* \text{ then} \\ \mathrm{len}(b) = p + 1 \text{ and } \mathrm{RP}(\tilde{z}, b) \leqslant p + 1 < \mathrm{RP}(z^*, b) \text{ for all } \tilde{z} \in [1, z] \\ \text{and } \mathrm{len}(b^*) = p + 2 \text{ and } b^*(1, p+1) = z < x = x^* = \infty \end{cases}$$

and

$$(12) \quad \begin{cases} \text{if } z < z^{**} \leqslant z^* \text{ and } p < t = t^* \text{ then } b'^{**} = b^{**} \\ \text{and } x'^* = x'^{**} = \infty = x = x^* \text{ and } x' = x^{**} \neq \infty. \end{cases}$$

Now if $z^{**} \leqslant z < z^*$ and $p < t = t^*$ then by (11) we see that $x^{**} < x = x^* = \infty$ and $\mathrm{RP}(z^{**}, b) \leqslant \mathrm{len}(b) < \mathrm{RP}(z^*, b)$ and hence upon taking (b, z^*, z^{**}) for (a, z, z^*) in (12) we get $b''^* = b^{**}$ and $x''^* = x^*$ and $x''^{**} = x^{**}$; therefore

$$(13) \quad \begin{cases} \text{if } z^{**} \leqslant z < z^* \text{ and } p < t = t^* \text{ then } b''^* = b^{**} \\ \text{and } x'' = x''^* = \infty = x = x^* \text{ and } x''^{**} = x^{**} \neq \infty. \end{cases}$$

Fourthly if $z < z^*$ and $t = t^* = p$ then

$$a(1, i) < \cdots < a(1, p-1) < z < z^* \leqslant x'^* = a(1, p)$$

and $\mathrm{len}(b) = p > 0$ and $x = x'^*$ and

$$b(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p-1] \\ z & \text{if } i = p \end{cases}$$

and $\text{len}(b'^*) = p$ and

$$b'^*(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p-1] \\ z^* & \text{if } i = p \end{cases}$$

and hence

$$\text{RP}(\tilde{z}, b) \leq p < p + 1 = \text{RP}(z^*, b) \text{ for all } \tilde{z} \in [1, z]$$

and $\text{len}(b^*) = p + 1$ and $x^* = \infty$ and

$$b^*(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p-1] \\ z & \text{if } i = p \\ z^* & \text{if } i = p + 1 \end{cases}$$

and

$$b^*(1, p) = z < z^* = b^*(1, p + 1) \leq x < x^* = \infty$$

and $\text{RP}(z, b'^*) = t = p = \text{len}(b')$ and $x' = z^*$ and

$$b'(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p-1] \\ z & \text{if } i = p \end{cases}$$

and therefore if also $z < z^{**} \leq z^*$ then: $\text{len}(b^{**}) = p + 1$ and $x^{**} = z^*$ and

$$b^{**}(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p-1] \\ z & \text{if } i = p \\ z^{**} & \text{if } i = p + 1 \end{cases}$$

and $\text{len}(b'^{**}) = p + 1$ and $x'^{**} = \infty$ and

$$b'^{**}(1, i) = \begin{cases} a(1, i) & \text{if } i = [1, p-1] \\ z & \text{if } i = p \\ z^{**} & \text{if } i = p + 1 \end{cases}$$

and hence $b'^{**} = b^{**}$. Thus

$$(14) \begin{cases} \text{if } z < z^* \text{ and } t = t^* = p \text{ then} \\ \text{len}(b) > 0 \text{ and } \text{RP}(\tilde{z}, b) \leq p < \text{RP}(z^*, b) \text{ for all } \tilde{z} \in [1, z] \\ \text{and } \text{len}(b^*) = p + 1 \text{ and } b^*(1, p) = z \leq x < x^* = \infty \end{cases}$$

and

$$(15) \begin{cases} \text{if } z < z^{**} \leq z^* \text{ and } t = t^* = p \text{ then } b'^{**} = b^{**} \\ \text{and } x'^{**} = \infty = x^* \text{ and } x \neq \infty \neq x^{**} \text{ and } (x'^*, x') = (x, x^{**}). \end{cases}$$

Now if $z^{**} \leq z < z^*$ and $t = t^* = p$ then by (14) we see that $x^{**} \leq x < x^* = \infty$ and $\text{RP}(z^{**}, b) \leq \text{len}(b) < \text{RP}(z^*, b)$ and hence upon taking (b, z^*, z^{**}) for (a, z, z^*) in (15) we get $b''^* = b^{**}$ and $x''^* = x^*$ and $x''^{**} = x^{**}$; therefore

$$(16) \begin{cases} \text{if } z^{**} \leq z < z^* \text{ and } t = t^* = p \text{ then } b''^* = b^{**} \\ \text{and } x''^* = \infty = x^* \text{ and } x \neq \infty \neq x^{**} \text{ and } (x'', x''^{**}) = (x, x^{**}). \end{cases}$$

Fifthly if $z < z^*$ and $t = t^* < p$ then

$$a(1, 1) < \cdots < a(1, t-1) < z < z^* \leq x'^* = a(1, t) < \cdots < a(1, p)$$

and $\operatorname{len}(b) = p$ and $x = x'^* < \infty$ and

$$b(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \setminus \{t\} \\ z & \text{if } i = t \end{cases}$$

and $\operatorname{len}(b'^*) = p$ and

$$b'^*(1, i) = \begin{cases} a(1, i) & \text{if } i = [1, p] \setminus \{t\} \\ z^* & \text{if } i = t \end{cases}$$

and hence

$$\mathrm{RP}(\tilde{z}, b) \leq t < t + 1 = \mathrm{RP}(z^*, b) \leq p \text{ for all } \tilde{z} \in [1, z]$$

and $\operatorname{len}(b^*) = p$ and $x^* = a(1, t+1)$ and

$$b^*(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \setminus \{t, t+1\} \\ z & \text{if } i = t \\ z^* & \text{if } i = t + 1 \end{cases}$$

and

$$b^*(1, t) = z < z^* \leq x = a(1, t) < a(1, t+1) = x^* < \infty$$

and $\mathrm{RP}(z, b'^*) = t$ and $\operatorname{len}(b') = p$ and $x' = z^*$ and

$$b'(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \setminus \{t\} \\ z & \text{if } i = t \end{cases}$$

and therefore if also $z < z^{**} \leq z^*$ then: $\operatorname{len}(b^{**}) = p$ and $x^{**} = z^*$ and

$$b^{**}(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \setminus \{t, t+1\} \\ z & \text{if } i = t \\ z^{**} & \text{if } i = t + 1 \end{cases}$$

and $\operatorname{len}(b'^{**}) = p$ and $x'^{**} = a(1, t+1)$ and

$$b'^{**}(1, i) = \begin{cases} a(1, i) & \text{if } i \in [1, p] \setminus \{t, t+1\} \\ z & \text{if } i = t \\ z^{**} & \text{if } i = t + 1 \end{cases}$$

and hence $b'^{**} = b^{**}$ and $x^{**} \leq x < x^* < \infty$ and $(x'^*, x', x'^{**}) = (x, x^{**}, x^*)$. Thus

$$(17) \quad \begin{cases} \text{if } z < z^* \text{ and } t = t^* < p \text{ then} \\ \operatorname{len}(b) = p \text{ and } \mathrm{RP}(\tilde{z}, b) < \mathrm{RP}(z^*, b) \leq p \text{ for all } \tilde{z} \in [1, z] \\ \text{and } \operatorname{len}(b^*) = p \text{ and } b^*(1, t) = z \leq x < x^* < \infty \end{cases}$$

and

$$(18)\ \begin{cases}\text{if } z<z^{**}\leqslant z^* \text{ and } t=t^*<p \text{ then } b'^{**}=b^{**}\\ \text{and } x^{**}\leqslant x<x^*<\infty \text{ and } (x'^*, x', x'^{**})=(x, x^{**}, x^*).\end{cases}$$

Now if $z^{**}\leqslant z<z^*$ and $t=t^*<p$ then by (17) we see that $x^{**}\leqslant x<x^*<\infty$ and $\mathrm{RP}(z^{**}, b)<\mathrm{RP}(z^*, b)\leqslant \mathrm{len}(b)$ and hence upon taking (b, z^*, z^{**}) for (a, z, z^*) in (18) we get $b''^*=b^{**}$ and $x''^*=x^*$ and $x''^{**}=x^{**}$; therefore

$$(19)\ \begin{cases}\text{if } z^{**}\leqslant z<z^* \text{ and } t=t^*<p \text{ then } b''^*=b^{**}\\ \text{and } x^{**}\leqslant x<x^*<\infty \text{ and } (x'', x''^{**}, x''^*)=(x, x^{**}, x^*).\end{cases}$$

If $z<z^*$ then clearly either $t<t^*\leqslant p$ or $t\leqslant p<t^*$ or $p<t=t^*$ or $t=t^*=p$ or $t=t^*<p$; therefore (′) and (″) follow from (4), (9), (12), (15), (18) and (5), (10), (13), (16), (19) respectively, where we note that these items respectively correspond to (1′), (3′), (3′), (3′), (2′) and (1″), (2″), (2″), (2″), (1″). □

4. Rodeletion from univectors

Definition 4.1. For discussing the inverse of roinsertion, given any $\hat{a}\in \mathrm{vec}(1)$, upon letting $\hat{p}=\mathrm{len}(\hat{a})$, we introduce the *rodeletoid* of $\hat{a}$ which we denote by $\mathrm{rode}(\hat{a})$ and which we define by putting

$$\mathrm{rode}(\hat{a})=\{\tilde{z}\in N^*: \hat{a}(1, u)\leqslant \tilde{z} \text{ for some } u\in [1, \hat{p}]\}$$

and we note that then

$$\mathrm{rode}(\hat{a})=\begin{cases}\emptyset & \text{if } \hat{p}=0\\ \{\tilde{z}\in N^*: \hat{a}(1, 1)\leqslant \tilde{z}\} & \text{if } \hat{p}\neq 0.\end{cases}$$

Now given any $\hat{z}\in \mathrm{rode}(\hat{a})$, first pictorially and then more precisely, we shall define certain elements $\mathrm{RDE}(\hat{z}, \hat{a})\in N^*$, $\mathrm{RDP}(\hat{z}, \hat{a})\in [1, \hat{p}]$, and $\mathrm{RD}(\hat{z}, \hat{a})\in \mathrm{vec}(1)$ which we may respectively call the *rodeletive entry* of $(\hat{z}, \hat{a})$, the *rodeletive place* of $(\hat{z}, \hat{a})$, and the *rodeletion* of $(\hat{z}, \hat{a})$. Pictorially we define

$$\mathrm{RDE}(\hat{z}, \hat{a})=\hat{x},\quad \mathrm{RDP}(\hat{z}, \hat{a})=\hat{t} \text{ and } \mathrm{RD}(\hat{z}, \hat{a})=\hat{b}$$

where

$$\begin{array}{llllll}\hat{a}(1,1)<\cdots< & \hat{a}(1,\hat{t}-1)<\hat{a}(1,\hat{t})=\hat{x}\leqslant\hat{z} & & <\hat{a}(1,\hat{t}+1)<\cdots & & <\hat{a}(1,\hat{p})\\ \| & \| & \| & \| & & \|\\ \hat{b}(1,1)<\cdots< & \hat{b}(1,\hat{t}-1) & <\hat{b}(1,\hat{t}) & <\hat{b}(1,\hat{t}+1)<\cdots & & <\hat{b}(1,\hat{p}).\end{array}$$

More precisely, we note that clearly there exists a unique $\hat{x}\in N^*$ and $\hat{t}\in [1, \hat{p}]$

and $\hat{b} \in \operatorname{vec}(1)$ such that

$$\begin{cases} \hat{a}(1, \hat{t}) = \hat{x} \leqslant \hat{z} \\ \text{and } \hat{a}(1, i) < \hat{x} \text{ for all } i \in [1, \hat{t}-1] \\ \text{and } \hat{z} < \hat{a}(i, j) \text{ for all } j \in [\hat{t}+1, \hat{p}] \end{cases}$$

and

$$\operatorname{len}(\hat{b}) = \hat{p}$$

and

$$\hat{b}(1, i) = \begin{cases} \hat{a}(1, i) & \text{if } i \in [1, \hat{p}] \setminus \{\hat{t}\} \\ \hat{z} & \text{if } i = \hat{t} \in [1, \hat{p}] \end{cases}$$

and we define $\operatorname{RDE}(\hat{z}, \hat{a}) \in N^*$ and $\operatorname{RDP}(\hat{z}, \hat{a}) \in [1, \hat{p}]$ and $\operatorname{RD}(\hat{z}, \hat{a}) \in \operatorname{vec}(1)$ by putting

$$\operatorname{RDE}(\hat{z}, \hat{a}) = \hat{x}, \quad \operatorname{RDP}(\hat{z}, \hat{a}) = \hat{t} \text{ and } \operatorname{RD}(\hat{z}, \hat{a}) = \hat{b}.$$

Lemma 4.2. *In the situation of* (4.1), *let* $\hat{a}' \in \operatorname{vec}(1)$ *be such that* $\hat{a}' \leqslant \hat{a}$. *Then we have the following.*

(4.2.1) *We have* $1 \leqslant \hat{t} \leqslant \operatorname{len}(\hat{a}) \leqslant \operatorname{len}(\hat{a}')$ *and* $\hat{a}'(1, \hat{t}) \leqslant \hat{a}(1, \hat{t}) = \hat{x} \in \operatorname{rode}(\hat{a}')$, *and upon letting* $\hat{t}' = \operatorname{RDP}(\hat{x}, \hat{a}')$ *we have* $\hat{t} \leqslant \hat{t}'$.

(4.2.2) *Upon letting* $\hat{b}' = \operatorname{RD}(\hat{x}, \hat{a}')$ *we have* $\hat{b}' \leqslant \hat{b}$.

Proof. Since $\hat{a}' \leqslant \hat{a}$, we get $1 \leqslant \hat{t} \leqslant \operatorname{len}(\hat{a}) \leqslant \operatorname{len}(\hat{a}')$ and $\hat{a}'(1, \hat{t}) \leqslant \hat{a}(1, \hat{t}) = \hat{x}$. Therefore $\hat{x} \in \operatorname{rode}(\hat{a}')$, and by the definition of $\hat{t}'$ we get $\hat{t} \leqslant \hat{t}'$. Now by the definitions of $\hat{b}$ and $\hat{b}'$ we get $\operatorname{len}(\hat{b}') = \operatorname{len}(\hat{a}') \geqslant \operatorname{len}(\hat{a}) = \operatorname{len}(\hat{b}) = \hat{p}$ and

$$\hat{b}(1, i) = \begin{cases} \hat{a}(1, i) \geqslant \hat{a}'(1, i) = \hat{b}'(1, i) & \text{if } i \in [1, \hat{p}] \setminus \{\hat{t}, \hat{t}'\} \\ \hat{a}(1, i) > \hat{x} = \hat{b}'(1, i) & \text{if } \hat{t} < \hat{t}' = i \in [1, \hat{p}] \\ \hat{z} \geq \hat{x} = \hat{b}'(1, i) & \text{if } \hat{t}' = \hat{t} = i \in [1, \hat{p}] \\ \hat{z} \geqslant \hat{x} \geqslant \hat{a}'(1, i) = \hat{b}'(1, i) & \text{if } \hat{t}' > \hat{t} = i \in [1, \hat{p}] \end{cases}$$

and hence $\hat{b}' \leqslant \hat{b}$. □

Lemma 4.3. *In the situation of* (3.1), *upon assuming that* $x \neq \infty$ *and upon letting* $\hat{z} = x$ *and* $\hat{a} = b$ *we have* $\hat{z} \in \operatorname{rode}(\hat{a})$, *and upon letting* $\hat{x}$, $\hat{t}$ *and* $\hat{b}$ *to be as in* (4.1) *we have*: $\hat{x} = z$, $\hat{t} = t$ *and* $\hat{b} = a$.

Proof. Obvious. □

Lemma 4.4. *In the situation of* (4.1), *upon letting* $z = \hat{x}$ *and* $a = \hat{b}$, *and upon letting* x, t *and* b *to be as in* (3.1) *we have*:
$x \neq \infty$, $x = \hat{z}$, $t = \hat{t}$ *and* $b = \hat{a}$.

Proof. Obvious. □

5. Roinsertion in unitableaux

Definition 5.1. Given any $T \in \text{stab}(1)$, upon letting $d = \text{dep}(T)$, we introduce the *roperiphery* of T which we denote by $\text{rope}(T)$ and which we define by putting

$$\text{rope}(T) = \{\bar{s} \in [1, d]: \text{len}(T[\bar{s}]) > \text{len}(T[e]) \text{ for all } e \in [\bar{s}+1, d]\}.$$

Now given any $z \in N^*$, firstly by (3.3.1) we see that there exists a unique $s \in [1, d+1]$ together with $x \in \text{covec}[1, s]$ and $t \in \text{acovec}[1, s]$ such that

$$\begin{cases} x(1,1) = z, \\ \text{and } x(1, e+1) = \text{RE}(x(1, e), [T, e]) \text{ for all } e \in [1, s-1], \\ \text{and } \text{RE}(x(1, s), [T, s]) = \infty, \\ \text{and } t(1, e) = \text{RP}(x(1, e), [T, e]) \text{ for all } e \in [1, s], \end{cases}$$

and we define $\text{RG}(z, T) \in N^*$, $\text{RE}(z, T) \in \text{covec}(1)$, $\text{RGE}(z, T) \in N^*$, $\text{RP}(z, T) \in \text{acovec}(1)$, and $\text{RGP}(z, T) \in N^*$ by putting $\text{RG}(z, T) = s$, $\text{RE}(z, T) = x$, $\text{RGE}(z, T) = x(1, s)$, $\text{RP}(z, T) = t$, and $\text{RGP}(z, T) = t(1, s)$, and we remark that $\text{RG}(z, T)$, $\text{RE}(z, T)$, $\text{RGE}(z, T)$, $\text{RP}(z, T)$, and $\text{RGP}(z, T)$ may respectively be called the *roinsertive tag* of (z, T), the *roinsertive entry* of (z, T), the *roinsertive tagged entry* of (z, T), the *roinsertive place* of (z, T), and the *roinsertive tagged place* of (z, T), and secondly by (3.3.2) we see that there exists a unique $U \in \text{stab}(1)$ such that

$$\text{dep}(U) = \begin{cases} d & \text{if } s \in [1, d] \\ d+1 & \text{if } s = d+1 \end{cases}$$

and

$$U[e] = \begin{cases} R(x(1, e), [T, e]) & \text{if } e \in [1, s] \\ [T, e] & \text{if } e \in [1, \text{dep}(U)] \setminus [1, s] \end{cases}$$

and we define $R(z, T) \in \text{stab}(1)$ by putting $R(z, T) = U$ and we remark that $R(z, T)$ may be called the *roinsertion* of (z, T), and we note that clearly

$$s \in \text{rope}(U) \text{ and } t(1, s) = \text{len}(U[s]) \text{ and } x(1, s) = U[s](1, \text{len}(U[s])).$$

Definition 5.2. Given any $w \in \text{popre}(1)$ and $T \in \text{stab}(1)$, upon letting $r = \text{len}(w)$, firstly we note that there exists a unique sequence $B = B(i)_{1 \le i \le r+1}$, with $B(1), \ldots, B(r+1)$ in $\text{stab}(1)$, such that $B(1) = T$ and $B(i+1) = R(w(1, i), B(i))$ for all $i \in [1, r]$, and we define $\text{RQ}(w, T)$ by putting $\text{RQ}(w, T) = B$, and remark that $\text{RQ}(w, T)$ may be called the *sequential roinsertion* of (w, T), and secondly we define $R(w, T) \in \text{stab}(1)$ by putting $R(w, T) = B(r+1)$ and we remark that $R(w, T)$ may be called the *roinsertion* of (w, T).

Again given any $w \in \text{popre}(1)$, we define $R(w) \in \text{stab}(1)$ by putting $R(w) = R(w, \text{stab}_0(1))$ and we remark that $R(w)$ may be called the *roinsertion* of w.

Lemma 5.3. *In the situation of* (5.1), *given any* $z^* \in N^*$, *upon letting* $s^* = \mathrm{RG}(z^*, U)$, $x^* = \mathrm{RE}(z^*, U)$, *and* $t^* = \mathrm{RP}(z^*, U)$, *we have the following.*

(5.3.1) $z^* \leqslant z \Rightarrow$ *for all* $e \in [1, s]$ *we have* $s^* > e$ *and* $t^*(1, e) \leqslant t(1, e)$ *and* $x^*(1, e) \leqslant x(1, e)$.

(5.3.2) $z < z^* \Rightarrow$ *for all* $e \in [1, s^*]$ *we have* $s \geqslant e$ *and* $t(1, e) < t^*(1, e)$ *and* $x(1, e) < x^*(1, e)$.

(5.3.3) $z^* \leqslant z \Leftrightarrow s^* > s \Leftrightarrow t^*(1, s^*) \leqslant t(1, s)$.

Proof. By taking $x(1, e)$, $[T, e]$, $x^*(1, e)$, $[U, e]$ for z, a, z^*, b in (3.4.1) we see that: if $e \in [1, s]$ is such that $s^* \geqslant e$ and $x^*(1, e) \leqslant x(1, e)$ then $s^* > e$ and $t^*(1, e) \leqslant t(1, e)$, and, moreover, if also $s > e$ then $x^*(1, e+1) \leqslant x(1, e+1)$; therefore by induction on e we get (5.3.1). By taking $x(1, e)$, $[T, e]$, $x^*(1, e)$, $[U, e]$ for z, a, z^*, b in (3.4.2) and (3.4.3) we see that: if $e \in [1, s^*]$ is such that $s \geqslant e$ and $x(1, e) < x^*(1, e)$ then $t(1, e) < t^*(1, e)$ and, moreover, if also $s^* > e$ then $s > e$ and $x(1, e+1) < x^*(1, e+1)$; therefore by induction on e we get (5.3.2). Since $t^* \in \mathrm{acovec}(1)$, we know that if $s^* > s$ then $t^*(1, s^*) \leqslant t^*(1, s)$ and hence by taking $e = s$ in (5.3.1) we see that: $z^* \leqslant z \Rightarrow s^* > s$ and $t^*(1, s^*) \leqslant t(1, s)$; similarly, since $t^* \in \mathrm{acovec}(1)$, we know that if $s \geqslant s^*$ then $t(1, s) \leqslant t(1, s^*)$ and hence by taking $e = s^*$ in (5.3.2) we see that: $z < z^* \Rightarrow s \geqslant s^*$ and $t(1, s) < t^*(1, s^*)$; now (5.3.3) follows from these two implications. □

Lemma 5.4. *For every* $w \in \mathrm{popre}(1)$ *we have* $\mathrm{con}(R(w)) = \mathrm{con}[w]$.

Proof. Obvious. □

Lemma 5.5. *In the situation of* (5.1) *we have*

$$\mathrm{dep}(U) > 0 \quad \textit{and} \quad U[1] = R(z, [T, 1]) = \begin{cases} R(z, T[1]) & \textit{if } d > 0 \\ R(z, \mathrm{vec}_0(1)) & \textit{if } d = 0 \end{cases}$$

and in the situation of (5.1) *and* (5.2) *we have*

$$\mathrm{dep}(R(w, T)) = 0 \Leftrightarrow r = 0 = d$$

and in the situation of (5.2) *we have*

$$[R(w, T), 1] = R(w, [T, 1]).$$

Proof. Obvious. □

Lemma 5.6. *Given any* $w \in \mathrm{popre}(1)$, *upon letting* $r = \mathrm{len}(w)$, *we have*

$$\mathrm{dep}(R(w)) = 0 \Leftrightarrow r = 0$$

and we have

$$\mathrm{RL}(w) = [R(w), 1] = \begin{cases} R(w)[1] & \textit{if } r > 0 \\ \mathrm{vec}_0(1) & \textit{if } r = 0 \end{cases}.$$

Proof. Obvious. □

Theorem 5.7. *For every* $w \in \text{popre}(1)$ *we have* $[R(w), 1] = \text{veg}(w)$, *and hence in particular we have* $\text{len}([R(w), 1]) = \text{inc}(w)$.

Proof. Follows from (3.6) and (5.6). $\square$

Lemma 5.8. *For every* $T \in \text{stab}(1)$ *we have* $R(\text{covas}(T)) = T$.

Proof. Let $d = \text{len}(T)$. The assertion being obvious when $d = 0$, henceforth assume that $d \neq 0$. For every $f \in [1, d]$ and $j \in [1, \text{len}((T, f))]$, upon letting

$$H(f, j) = j + \sum_{e \in [1, f-1]} \text{len}((T, w))$$

we get $v[f, j] \in \text{popre}[1, H(f, j)]$ by putting

$$v[f, j](1, i) = \text{covas}(T)(1, i) \text{ for all } i \in [1, H(f, j)]$$

and we get $T(f, j) \in \text{stab}(1)$ with $\text{len}(T(f, j)) = f$ by putting

$$(T(f, j), e) = (T, e) \text{ for all } e \in [1, f-1]$$

and

$$\text{len}((T(f, j), f)) = j$$

and

$$(T(f, j), f)(1, i) = (T, f)(1, \text{len}((T, f)) - j + i) \text{ for all } i \in [1, j].$$

Now given any $f \in [2, d]$ and $j \in [1, \text{len}((T, f))]$, upon letting

$$(f^*, j^*) = \begin{cases} (f, j-1) & \text{if } j > 1 \\ (f-1, \text{len}((T, f-1)) & \text{if } j = 1, \end{cases}$$

firstly we have

$$\text{dep}(T(f^*, j^*)) = \text{dep}(T(f, j)) = \text{dep}(T) \geqslant j \geqslant 1$$

and

$$\text{len}(T(f, j)[1]) = f \text{ and } T(f, j)[1](1, f) = v[f, j](1, H(f, j))$$

and secondly for every $e \in [1, j-1]$ we have

$$\text{len}(T(f^*, j^*)[e]) = f = \text{len}(T(f, j)[e])$$

and

$$T(f^*, j^*)[e](1, i) = T(f, j)[e](1, i) \text{ for all } i \in [1, f-1]$$

and

$$T(f^*, j^*)[e](1, f-1) < T(f, j)[e](1, f) \leqslant T(f^*, j^*)[e](1, f)$$

and

$$T(f^*, j^*)[e](1, f) = T(f, j)[e+1](1, f)$$

and thirdly for $e = j$ we have

$$\text{len}(T(f^*, j^*)[e]) = f - 1 \text{ and } \text{len}(T(f, j)[e]) = f$$

and

$$T(f^*, j^*)[e](1, i) = T(f, j)[e](1, i) \text{ for all } i \in [1, f-1]$$

and

$$T(f^*, j^*)[e](1, f-1) < T(f, j)[e](1, f)$$

and fourthly for every $e \in [j+1, \mathrm{dep}(T)]$ we have

$$T(f^*, j^*)[e] = T(f, j)[e]$$

and hence we get

$$R(v[f, j](1, H(f, j)), T(f^*, j^*)) = T(f, j).$$

Next, given any $j \in [2, \mathrm{len}((T, 1))]$, firstly we have

$$1 + \mathrm{dep}(T(1, j-1)) = \mathrm{dep}(T(1, j)) = j \geqslant 2$$

and

$$\mathrm{len}(T(1, j)[1]) = 1 \text{ and } T(1, j)[1](1, 1) = v[1, j](1, H(1, j))$$

and secondly for every $e \in [1, \mathrm{dep}(T(1, j)) - 1]$ we have

$$\mathrm{len}(T(1, j-1)[e]) = 1 = \mathrm{len}(T(1, j)[e])$$

and

$$T(1, j)[e](1, 1) \leqslant T(1, j-1)[e](1, 1) = T(1, j)[e+1](1, 1)$$

and thirdly for $e = \mathrm{dep}(T(1, j))$ we have

$$\mathrm{len}(T(1, j)[e]) = 1$$

and hence

$$R(v[1, j](1, H(1, j)), T(1, j-1)) = T(1, j).$$

Thus for any $f \in [1, d]$ and $j \in [1, \mathrm{len}((T, f))]$ we have

$$T(f, j) = \begin{cases} R(v[f, j](1, H(f, j)), T(f, j-1)) & \text{if } j > 1 \\ R(v[f, 1](1, H(f, 1)), T(f+1, \mathrm{len}(T[f+1]))) & \text{if } j = 1 \text{ and } f > 1 \end{cases}$$

and obviously we have

$$T(1, 1) = R(v[1, 1](1, H(1, 1)), \mathrm{stab}_0(1))$$

and therefore by induction on $H(f, j)$ we see that for all $f \in [1, d]$ and $j \in [1, \mathrm{len}((T, f))]$ we have $R(v[f, j]) = T(f, j)$. Now upon taking $f = d$ and $j = \mathrm{len}((T, d))$, and upon noting that $v[d, \mathrm{len}((T, d))] = \mathrm{covas}(T)$ and $T(d, \mathrm{len}((T, d))) = T$, we get $R(\mathrm{covas}(T)) = T$. $\square$

Lemma 5.9. *For every* $T \in \mathrm{stab}(1)$ *we have* $R(\mathrm{vas}(T)) = T$.

Proof. Let $d = \mathrm{dep}(T)$. The assertion being obvious when $d = 0$, henceforth assume that $d \neq 0$. For every $f \in [1, d]$ and $j \in [1, \mathrm{len}(T[f])]$, clearly there exists a unique $g(f, j) \in [1, d-f+1]$ such that

$$\begin{cases} \mathrm{len}(T[f+e-1]) \geq j & \text{for all } e \in [1, g(f, j)] \\ \mathrm{len}(T[f+e-1]) < j & \text{for all } e \in [g(f, j)+1, d-f+1] \end{cases}$$

and upon letting

$$H(f, j) = j + \sum_{e \in [f+1, d]} \mathrm{len}(T[e])$$

we get $v[f, j] \in \mathrm{popre}[1, H(f, j)]$ by putting

$$v[f, j](1, i) = \mathrm{vas}(T)(1, i) \text{ for all } i \in [1, H(f, j)]$$

and we get $T(f, j) \in \mathrm{stab}(1)$ with $\mathrm{dep}(T(f, j)) = d - f + 1$ by putting

$$\mathrm{len}(T(f, j)[e]) = \begin{cases} \mathrm{len}(T[f+e]) & \text{if } e \in [1, g(f, j)-1] \\ j & \text{if } e = g(f, j) \\ \mathrm{len}(T[f+e-1]) & \text{if } e = [g(f, j)+1, d-f+1] \end{cases}$$

and

$$T(f, j)[e](1, i) = \begin{cases} T[f+e-1](1, i) & \text{if } e \in [1, g(f, j)] \text{ and } i \in [1, j] \\ T[f+e](1, i) & \text{if } e \in [1, g(f, j)-1] \\ & \text{and } i \in [j+1, \mathrm{len}(T(f, j)[e])] \\ T[f+e-1](1, i) & \text{if } e \in [g(f, j)+1, d-f+1] \\ & \text{and } i \in [1, \mathrm{len}(T(f, j)[e])]. \end{cases}$$

Now given any $f \in [1, d]$ and $j \in [2, \mathrm{len}(T[f])]$, firstly we have

$$\mathrm{dep}(T(f, j-1)) = \mathrm{dep}(T(f, j)) = d - f + 1 \geq g(f, j) \geq 1$$

and

$$\mathrm{len}(T(f, j)[1]) \geq j \text{ and } T(f, j)[1](1, j) = v[f, j](1, H(f, j))$$

and secondly for every $e \in [1, g(f, j)-1]$ we have

$$\mathrm{len}(T(f, j-1)[e]) = \mathrm{len}(T(f, j)[e])$$

and

$$T(f, j-1)[e](1, i) = T(f, j)[e](1, i) \text{ for all } i \in [1, \mathrm{len}(T(f, j)[e])] \setminus \{j\}$$

and

$$T(f, j-1)[e](1, j-1) < T(f, j)[e](1, j) \leq T(f, j-1)[e](1, j)$$

and

$$T(f, j-1)[e](1, j) = T(f, j)[e+1](1, j)$$

and thirdly for $e = g(f, j)$ we have

$$\mathrm{len}(T(f, j-1)[e]) = j - 1 \text{ and } \mathrm{len}(T(f, j)[e]) = j$$

and

$$T(f, j-1)[e](1, i) = T(f, j)[e](1, i) \text{ for all } i \in [1, j-1]$$

and

$$T(f, j-1)[e](1, j-1) < T(f, j)[e](1, j)$$

and fourthly for every $e \in [g(f, j)+1, d-f+1]$ we have

$$T(f, j-1)[e] = T(f, j)[e]$$

and hence we get

$$R(v[f, j](1, H(f, j)), T(f, j-1)) = T(f, j).$$

Next, given any $f \in [1, d-1]$, upon letting $f^* = f+1$ and $j^* = \text{len}(T[f^*])$, firstly we have

$$1 + \text{dep}(T(f^*, j^*)) = \text{dep}(T(f, 1)) = d-f+1 = g(f, 1) \geq 2$$

and

$$\text{len}(T(f, 1)[1]) \geq 1 \text{ and } T(f, 1)[1](1, 1) = v[f, 1](1, H(f, 1))$$

and secondly for every $e \in [1, \text{dep}(T(f, 1))-1]$ we have

$$\text{len}(T(f^*, j^*)[e]) = \text{len}(T(f, 1)[e])$$

and

$$T(f^*, j^*)[e](1, i) = T(f, 1)[e](1, i) \text{ for all } i \in [1, \text{len}(T(f, 1)[e])] \setminus \{1\}$$

and

$$T(f, 1)[e](1, 1) \leq T(f^*, j^*)[e](1, 1) = T(f, 1)[e+1](1, 1)$$

and thirdly for $e = \text{dep}(T(f, 1))$ we have

$$\text{len}(T(f, 1)[e]) = 1$$

and hence

$$R(v[f, 1](1, H(f, 1)), T(f^*, j^*)) = T(f, 1).$$

Thus for any $f \in [1, d]$ and $j \in [1, \text{len}(T[f])]$ we have

$$T(f, j) = \begin{cases} R(v[f, j](1, H(f, j)), T(f, j-1)) & \text{if } j > 1 \\ R(v[f, 1](1, H(f, 1)), T(f+1, \text{len}(T[f+1]))) & \text{if } j = 1 \text{ and } f < d \end{cases}$$

and obviously we have

$$T(d, 1) = R(v[d, 1](1, H(d, 1)), \text{stab}_0(1))$$

and therefore by induction on $H(f, j)$ we see that for all $f \in [1, d]$ and $j \in [1, \text{len}(T[f])]$ we have $R(v[f, j]) = T(f, j)$. Now upon taking $f = 1$ and $j = \text{len}(T)$, and upon noting that $v[1, \text{len}(T)] = \text{vas}(T)$ and $T(1, \text{len}(T)) = T$, we get $R(\text{vas}(T)) = T$. □

Lemma 5.10. *For every $w \in \text{popre}(1)$ we have that $\text{vas}(R(w))$ is equivalent to w.*

Proof. Let $r = \text{len}(w)$ and let the situation be as in (5.1). By successively taking $(T[1], x(1, 1))$, $(T[2], x(1, 2)), \ldots, (T[s-1], x(1, s-1))$ for (a, z) in (3.7) we see that

$$(1)\ \begin{cases} \text{if } w^* \in \text{pre}[1, \text{are}(T)+1] \text{ is such that } w^*(1, i) = \text{vas}(T)(1, i) \\ \text{for all } i \in [1, \text{are}(T)] \text{ and } w^*(1, \text{are}(T)+1) = z, \\ \text{then vas}(U) \text{ is equivalent to } w^*. \end{cases}$$

If $r > 0$ then, upon letting $w' \in \text{popre}[1, r-1]$ with $w'(1, i) = w(1, i)$ for all $i \in [1, r-1]$ and upon taking $T = R(w')$ and $z = w(1, r)$, we clearly get $U = R(w)$, and hence with w^* as in (1), by (1) we see that $\text{vas}(R(w))$ is equivalent to w^*; now if $\text{vas}(R(w'))$ is equivalent to w' then clearly w^* is equivalent to w and hence, because "equivalence" is an equivalence relation, $\text{vas}(R(w))$ is equivalent to w; thus

$$(2)\ \begin{cases} \text{if } r > 0 \text{ and, upon letting } w' \in \text{popre}[1, r-1] \text{ with} \\ w'(1, i) = w(1, i) \text{ for all } i \in [1, r-1], \text{ we have that vas}(R(w')) \\ \text{is equivalent to } w', \text{ then vas}(R(w)) \text{ is equivalent to } w. \end{cases}$$

If $r = 0$ then obviously $\text{vas}(R(w))$ is equivalent to w and hence, in view of (2), by induction on $\text{len}(w)$ we see that $\text{vas}(R(w))$ is always equivalent to w. □

Lemma 5.11. *In the situation of* (5.2), *if* $w^* \in \text{popre}(1)$ *is equivalent to* w *then* $R(w^*, T) = R(w, T)$.

Proof. Clearly we have a unique $T' \in \text{stab}(1)$ such that $[T', e] = [T, e+1]$ for all $e \in N^*$. Let $a = [T, 1]$ and $w' = \text{REF}(w, a)$, and given any $w^* \in \text{popre}(1)$ let $w'^* = \text{REF}(w^*, a)$. Now by (3.8) we see that

$$(1)\ \begin{cases} \text{if } w^* \text{ is equivalent to } w \\ \text{then } R(w, a) = R(w^*, a) \text{ and } w'^* \text{ is equivalent to } w' \end{cases}$$

and obviously

$$(2)\ [R(w, T), e] = \begin{cases} R(w, a) & \text{if } e = 1 \\ [R(w', T'), e-1] & \text{if } e \in N^* \setminus \{1\} \end{cases}$$

and

$$(3)\ [R(w^*, T), e] = \begin{cases} R(w^*, a) & \text{if } e = 1 \\ [R(w'^*, T'), e-1] & \text{if } e \in N^* \setminus \{1\} \end{cases}$$

and

$$(4)\ \begin{cases} \text{if dep}(T) = 0 = \text{len}(w) \text{ and } w^* \text{ is equivalent to } w \\ \text{then len}(w^*) = 0 \text{ and } R(w, T) = \text{stab}_0(1) = R(w^*, T) \end{cases}$$

and

(5) if $\text{dep}(T) = 0$ then $\text{dep}(T') = 0$

and

(6) if $\text{len}(w) > 0 = \text{dep}(T)$ then $\text{len}(w') < \text{len}(w)$

and

(7) if $\mathrm{dep}(T) > 0$ then $\mathrm{dep}(T') < \mathrm{dep}(T)$.

In view of (1) to (6), by induction on $\mathrm{len}(w)$ we see that

(8) if $\mathrm{dep}(T) = 0$ and w^* is equivalent to w then $R(w^*, T) = R(w, T)$.

In view of (1), (2), (3), (7,) (8), by induction on $\mathrm{dep}(T)$ we see that if w^* is equivalent to w then $R(w^*, T) = R(w, T)$. □

Theorem 5.12. *For any w and w^* in* popre(1) *we have that*: *w^* is equivalent to w iff $R(w^*) = R(w)$.*

Proof. In view of (5.10), our assertion follows by taking $\mathrm{stab}_0(1)$ for T in (5.11). □

Corollary 5.13. *For every $w \in$* popre(1) *we have that* $\mathrm{covas}(R(w))$ *is equivalent to w.*

Proof. In view of (5.12), our assertion follows by taking $R(w)$ for T in (5.8). □

6. Rodeletion from unitableaux

Definition 6.1. Given any $\hat{T} \in \mathrm{stab}(1)$ and $\hat{s} \in \mathrm{rope}(\hat{T})$, upon letting $\hat{d} = \mathrm{dep}(\hat{T})$, firstly, by decreasing induction on e, in view of (4.2.1) we see that there exists a unique $\hat{x} \in \mathrm{covec}[1, \hat{s}]$ and $\hat{t} \in \mathrm{acovec}[1, \hat{s}]$ such that

$$\hat{t}(1, \hat{s}) = \mathrm{len}(\hat{T}[\hat{s}]) \text{ and } \hat{x}(1, \hat{s}) = \hat{T}[\hat{s}](1, \mathrm{len}(\hat{T}[\hat{s}]))$$

and such that for every $e \in [1, \hat{s} - 1]$ we have

$$\begin{cases} 1 \leq \hat{t}(1, e+1) \leq \mathrm{len}(\hat{T}[e+1]) \leq \mathrm{len}(\hat{T}[e]) \\ \text{and } \hat{T}[e](1, \hat{t}(1, e+1)) \leq \hat{T}[e+1](1, \hat{t}(1, e+1)) = \hat{x}(1, e+1) \in \mathrm{rode}(\hat{T}[e]) \\ \text{and } \hat{t}(1, e) = \mathrm{RDP}(\hat{x}(1, e+1), \hat{T}[e]) \in [1, \mathrm{len}(\hat{T}[e])] \\ \text{and } \hat{x}(1, e) = \mathrm{RDE}(\hat{x}(1, e+1), \hat{T}[e]) = \hat{T}[e](1, \hat{t}(1, e)) \end{cases}$$

and we define $\mathrm{RDE}(\hat{s}, \hat{T}) \in \mathrm{covec}[1, \hat{s}]$ and $\mathrm{RDP}(\hat{s}, \hat{T}) \in \mathrm{acovec}[1, \hat{s}]$ by putting $\mathrm{RDE}(\hat{s}, \hat{T}) = \hat{x}$ and $\mathrm{RDP}(\hat{s}, \hat{T}) = \hat{t}$, and we remark that $\mathrm{RDE}(\hat{s}, \hat{T})$ and $\mathrm{RDP}(\hat{s}, \hat{T})$ may respectively be called the *rodeletive entry* of $(\hat{s}, \hat{T})$ and the *rodeletive place* of $(\hat{s}, \hat{T})$, and secondly by (4.2.2) we see that there exists a unique $\hat{U} \in \mathrm{stab}(1)$ such that

$$\mathrm{dep}(\hat{U}) = \begin{cases} \hat{d} & \text{if } \mathrm{len}(\hat{T}[\hat{s}]) > 1 \\ \hat{d} - 1 & \text{if } \mathrm{len}(\hat{T}[\hat{s}]) = 1 \end{cases}$$

and

$$\hat{U}[e] = \begin{cases} \mathrm{RD}(\hat{x}(1, e+1), \hat{T}[e]) & \text{if } r \in [1, \hat{s}-1] \\ \hat{T}[e] & \text{if } e \in [1, \mathrm{dep}(\hat{U})] \setminus [1, \hat{s}] \end{cases}$$

and

$$\begin{cases} \mathrm{len}([\hat{U}, \hat{s}]) = \mathrm{len}(\hat{T}[\hat{s}]) - 1 \\ \text{and } [\hat{U}, \hat{s}](1, i) = \hat{T}[\hat{s}](1, i) \text{ for all } i \in [1, \mathrm{len}([\hat{U}, \hat{s}])] \end{cases}$$

and we define $\mathrm{RD}(\hat{s}, \hat{T}) \in \mathrm{stab}(1)$ by putting $\mathrm{RD}(\hat{s}, \hat{T}) = \hat{U}$, and we remark that $\mathrm{RD}(\hat{s}, \hat{T})$ may be called the *rodeletion* of $(\hat{s}, \hat{T})$.

Lemma 6.2. *In the situation of* (5.1), *upon letting* $\hat{T} = U$, $\hat{d} = \mathrm{dep}(\hat{T})$ *and* $\hat{s} = s$ *we have* $\hat{s} \in \mathrm{rope}(\hat{T})$, *and upon letting* x, $\hat{t}$ *and* $\hat{U}$ *to be as in* (6.1) *we have*: $\hat{x} = x$, $\hat{t} = t$ *and* $\hat{U} = T$.

Proof. Follows from (4.3).□

Lemma 6.3. *In the situation of* (6.1), *upon letting* $z = \hat{x}(1, 1)$ *and* $T = \hat{U}$, *and upon letting* s, x, t *and* U *to be as in* (5.1) *we have*: $s = \hat{s}$, $x = \hat{x}$, $t = \hat{t}$ *and* $U = \hat{T}$.

Proof. Follows from (4.4). □

Lemma 6.4. *In the situation of* (6.1), *given any* $\hat{s}^* \in \mathrm{rope}(\hat{U})$, *upon letting* $\hat{x}^* = \mathrm{RDE}(\hat{s}^*, \hat{U})$ *and* $\hat{t}^* = \mathrm{RDP}(\hat{s}^*, \hat{U})$, *we have the following.*

(6.4.1) $\hat{x}(1, 1) \leqslant \hat{x}^*(1, 1) \Rightarrow$ *for all* $e \in [1, \hat{s}^*]$ *we have* $\hat{s} > e$ *and* $\hat{t}(1, e) \leqslant \hat{t}^*(1, e)$ *and* $\hat{x}(1, e) \leqslant \hat{x}^*(1, e)$.

(6.4.2) $\hat{x}^*(1, 1) < \hat{x}(1, 1) \Rightarrow$ *for all* $e \in [1, \hat{s}]$ *we have* $\hat{s}^* \geqslant e$ *and* $\hat{t}^*(1, e) < \hat{t}(1, e)$ *and* $\hat{x}^*(1, e) < \hat{x}(1, e)$.

(6.4.3) $\hat{x}(1, 1) \leqslant \hat{x}^*(1, 1) \Leftrightarrow \hat{s} > \hat{s}^* \Leftrightarrow \hat{t}(1, \hat{s}) \leqslant \hat{t}^*(1, \hat{s}^*)$.

Proof. Follows from (5.3), (6.2) and (6.3). □

7. Roinsertion in bivectors

Throughout this section let there be fixed any $k \in [1, 2]$, and let $k' = 1$ or 2 according as $k = 2$ or 1.

Definition 7.1. By $\mathrm{leb}(k)$ we denote the set of all $w \in \mathrm{popre}(2)$ such that for every $i \in [1, \mathrm{len}(w) - 1]$ we have either: $w(k', i) = w(k', i+1)$ and $w(k, i) \geqslant w(k, i+1)$, or: $w(k', i) < w(k', i+1)$, and we remark that a member of $\mathrm{leb}(k)$ may be called a *lexical bivector of type* k. For every $r \in N$ we put

$$\mathrm{leb}[k, r] = \{w \in \mathrm{leb}(k) \colon \mathrm{len}(w) = r\}.$$

Definition 7.2. Given any $z \in N^*(2)$, by $\text{vins}(k, z)$ we denote the set of all $a \in \text{vec}(2)$ such that either: $\text{len}(a) \neq 0$ and $a(k', \text{len}(a)) = z(k')$ and $a(k, \text{len}(a)) \geqslant z(k)$, or: $a(k', i) < z(k')$ for all $i \in [1, \text{len}(a)]$, and we remark that a member of $\text{vins}(k, z)$ may be called a *vectorial insertible* of (k, z), and we remark that obviously $\text{vec}_0(2) \in \text{vins}(k, z)$. Note that for any $w \in \text{leb}(k)$ we clearly have

$$\text{vins}(k, w[i]) \subset \text{vins}(k, w[j]) \text{ for all } i > j \text{ in } [1, \text{len}(w)].$$

Definition 7.3. Given any $z \in N^*(2)$ and $a \in \text{vins}(k, z)$, clearly there exists a unique $b \in \text{vins}(k, z)$ such that $k(b) = R(z(k), k(a))$ and

$$b(k', i) = \begin{cases} a(k', i) & \text{if } i \in [1, \text{len}(a)] \\ z(k') & \text{if } i \in [1, \text{len}(b)] \setminus [1, \text{len}(a)] \end{cases}$$

and we define $R(k, z, a) \in \text{vins}(k, z)$ by putting $R(k, z, a) = b$ and we remark that $R(k, z, a)$ may be called the *roinsertion* of (k, z, a).

Definition 7.4. For any $w \in \text{popre}(2)$ we put

$$\text{vins}(k, w) = \begin{cases} \text{vec}(2) & \text{if } \text{len}(w) = 0 \\ \text{vins}(k, w[1]) & \text{if } \text{len}(w) \neq 0 \end{cases}$$

and

$$\text{vits}(k, w) = \begin{cases} \text{vec}(2) & \text{if } \text{len}(w) = 0 \\ \text{vins}(k, w)[\text{len}(w)] & \text{if } \text{len}(w) \neq 0 \end{cases}$$

and we remark that a member of $\text{vins}(k, w)$ (resp: $\text{vits}(k, w)$) may be called a *vectorial insertible* (resp: *vectorial terminable*) of (k, w).

Definition 7.5. Given any $w \in \text{leb}(k)$ and $a \in \text{vins}(k, w)$, upon letting $r = \text{len}(w)$, clearly there exists a unique sequence $A = A(i)_{1 \leqslant i \leqslant r+1}$, with $A(i) \in \text{vins}(k, w[i])$ for all $i \in [1, r]$ and $A(r+1) \in \text{vits}(k, w)$, such that $A(1) = a$ and $A(i+1) = R(k, w[i], A(i))$ for all $i \in [1, r]$, and we define $\text{RQ}(k, w, a)$ by putting $\text{RQ}(k, w, a) = A$ and we remark that $\text{RQ}(k, w, a)$ may be called the *sequential roinsertion* of (k, w, a), and we define $R(k, w, a) \in \text{vits}(k, w)$ by putting $R(k, w, a) = A(r+1)$ and we remark that $R(k, w, a)$ may be called the *roinsertion* of (k, w, a).

Definition 7.6. Given any $w \in \text{leb}(k)$, we define $\text{RL}(k, w) \in \text{vits}(k, w)$ by putting $\text{RL}(k, w) = R(k, w, \text{vec}_0(2))$ and we remark that $\text{RL}(k, w)$ may be called the *vectorial roinsertion* of (k, w).

Lemma 7.7. *Given any $w \in \text{leb}(k)$, upon letting $r = \text{len}(w)$ and $a = \text{veg}(w)$, we have the following.*

(7.7.1) $\text{suve}(w) = \text{suve}(k(w))$ *and* $k(S[w]) = S[k(w)]$ *for all* $S \in \text{suve}(w)$.

(7.7.2) $\text{veg}(k(w)) = k(a)$.

(7.7.3) $a \in \text{vits}(k, w)$.

(7.7.4) *If* $y \in \text{leb}[k, r+1]$ *is such that* $y[i] = w[i]$ *for all* $i \in [1, r]$, *then upon letting* $z = y[r+1]$ *we have* $a \in \text{vins}(k, z)$ *and upon letting* $b = R(k, z, a)$ *we have* $b = \text{veg}(y)$.

Proof. (7.7.1) is obvious and, in view of (2.1), it yields (7.7.2). Let $p = \text{len}(a)$. If $r = 0$ then clearly $p = 0$ and hence $a \in \text{vits}(k, w)$ and in the situation of (7.7.4) we have $a \in \text{vins}(k, z)$ and $b = y = \text{veg}(y)$. So henceforth assume that $r > 0$ and note that then $0 < p \leq r$. For every $i \in N$ let

$$W(i) = \{S \in \text{suve}(w): \text{card}(S) = i\}.$$

Now

$$W(p) \neq \emptyset = W(i) \text{ for all } i \in N^* \setminus [1, p]$$

and

$$\begin{aligned} w(k', r) &= \max\{w(k', j): j \in [1, r]\} \text{ because } w \in \text{leb}(k) \\ &\geq \max\{S[w](k', p): S \in W(p)\} \\ &\geq \min\{S[w](k', p): S \in W(p)\} \\ &= a(k', p) \\ &\geq a(k', i) \text{ for all } i \in [1, p] \end{aligned}$$

and hence in case of $w(k', r) = a(k', p)$ we would have

$$\begin{aligned} w(k', r) = a(k', p) &\Rightarrow S[w](k', p) = w(k', r) \text{ for all } S \in W(p) \\ &\Rightarrow S[w](k, p) \geq w(k, r) \text{ for all } S \in W(p) \text{ because } w \in \text{leb}(k) \\ &\Rightarrow w(k, r) \leq \min\{S[w](k, p): S \in W(p)\} = a(k, p). \end{aligned}$$

Therefore $a \in \text{vins}(k, w[r])$ and hence $a \in \text{vits}(k, w)$ which proves (7.7.3).

To prove (7.7.4) let there be given any $y \in \text{leb}[k, r+1]$ such that $y[i] = w[i]$ for all $i \in [1, r]$, and let $z = y[r+1]$. By (7.7.3) it follows that $a \in \text{vins}(k, z)$. Now, let $c = \text{veg}(y)$. For every $i \in N$ let

$$\hat{W}(i) = \{S \in W(i): w(\tilde{k}, s) < z(\tilde{k}) \text{ for all } \tilde{k} \in [1, 2] \text{ and } s \in S\}$$

and

$$Y(i) = \{S \in \text{suve}(y): \text{card}(S) = i\}$$

and note that, given any $S \in \hat{W}(i)$, upon letting $S^* = S \cup \{r+1\}$, we have $S^* \in Y(i+1)$ and

$$S^*[y](\tilde{k}, j) = \begin{cases} S[w](\tilde{k}, i) & \text{if } \tilde{k} \in [1, 2] \text{ and } j \in [1, i] \\ z(\tilde{k}) & \text{if } \tilde{k} \in [1, 2] \text{ and } j = i+1. \end{cases}$$

Now clearly

$$W(0) = \{\emptyset\} = Y(0) \text{ and } W(i) = \emptyset \text{ for all } i \in N^* \setminus [1, p]$$

and

$$Y(i+1) = W(i+1) \cup \{S^*: S \in \hat{W}(i)\} \text{ for all } i \in N.$$

and by (2.5) we have

$$\begin{cases} W(i) \neq \emptyset \text{ and } a(k', i) = \min\{S[w](k', i) \colon S \in W(i)\} \\ \text{for all } i \in [1, p] \end{cases}$$

and

$$\begin{cases} Y(i) \neq \emptyset \text{ and } c(k', i) = \min\{S[y](k', i) \colon S \in Y(i)\} \\ \text{for all } i \in [1, \operatorname{len}(c)] \end{cases}$$

and, since $y \in \operatorname{leb}(k)$, we also have

$$w(k', 1) \leqslant w(k', 2) \leqslant \cdots \leqslant w(k', p) \leqslant z(k')$$

and hence it follows that

$$\begin{cases} \operatorname{len}(c) \in [p, p+1] \text{ and } c(k', i) = a(k', i) \text{ for all } i \in [1, p] \\ \text{and if } \operatorname{len}(c) = p+1 \text{ then } c(k', p+1) = z(k') \end{cases}$$

and by (7.7.2) and (3.6) we also see that

$$k(c) = k(b)$$

and therefore we must have $c = b$. $\square$

Theorem 7.8. *For every* $w \in \operatorname{leb}(k)$ *we have*

$$k(\mathrm{RL}(k, w)) = \mathrm{RL}(k(w)) = \operatorname{veg}(k(w)) = k(\operatorname{veg}(w))$$

and

$$\mathrm{RL}(k, w) = \operatorname{veg}(w) \textit{ and } \operatorname{len}(\mathrm{RL}(k, w)) = \operatorname{inc}(w) = \operatorname{inc}(k(w)).$$

Proof. Obviously $k(\mathrm{RL}(k, w)) = \mathrm{RL}(k(w))$, and by (5.7) we have $\mathrm{RL}(k(w)) = \operatorname{veg}(k(w))$ and $\operatorname{len}(\mathrm{RL}(k(w)) = \operatorname{inc}(k(w))$, and hence we get $\operatorname{len}(\mathrm{RL}(k, w) = \operatorname{inc}(k(w))$. If $\operatorname{len}(w) = 0$, i.e. if $w = \operatorname{vec}_0(2)$, then obviously $\mathrm{RL}(k, w) = w = \operatorname{veg}(w)$. Therefore, in view of (7.7), by induction on $\operatorname{len}(w)$ we get $\mathrm{RL}(k, w) = \operatorname{veg}(w)$ and hence, in view of (2.6), we get $\operatorname{len}(\mathrm{RL}(k, w)) = \operatorname{inc}(w)$. Now it also follows that $\operatorname{veg}(k(w)) = k(\operatorname{veg}(w))$. $\square$

8. Roinsertion in bitableaux

Throughout this section let there be fixed any $k \in [1, 2]$, and let $k' = 1$ or 2 according as $k = 2$ or 1.

Definition 8.1. Given any $z \in N^*(2)$, firstly by $\operatorname{led}(k, z)$ we denote the set of all $\bar{z} \in N^*(2)$ such that either: $\bar{z}(k') = z(k')$ and $\bar{z}(k) \leqslant z(k)$, or: $\bar{z}(k') > z(k')$, and we remark that a member of $\operatorname{led}(k, z)$ may be called a *lexical dominator* of z, and secondly by $\operatorname{pretins}(k, z)$ we denote the set of all $T \in \operatorname{stab}(2)$ such that for every

$e \in [1, \mathrm{dep}(T)]$ we have either: $T[e](k', \mathrm{len}(T[e])) = z(k')$ and $e < \mathrm{RG}(z(k), k(T))$, or: $T[e](k', \mathrm{len}(T[e])) < z(k')$, and we remark that a member of pretins(k, z) may be called a *tabular preinsertible* of (k, z), and thirdly by tins(k, z) we denote the set of all $T \in \mathrm{stab}(2)$ such that $T \in \mathrm{pretins}(k, \bar{z})$ for all $\bar{z} \in \mathrm{led}(k, z)$, and we remark that a member of tins(k, z) may be called a *tabular insertible* of (k, z), and we note that clearly $\mathrm{stab}_0(2) \in \mathrm{tins}(k, z)$.

Definition 8.2. Given any $z \in N^*(2)$ and $T \in \mathrm{pretins}(k, z)$, upon letting

$$s = \mathrm{RG}(z(k), k(T)), \quad x = \mathrm{RE}(z(k), k(T)) \text{ and } t = \mathrm{RP}(z(k), k(T))$$

firstly by (5.3.3) we see that there exists a unique $U \in \mathrm{tins}(k, z)$ such that $k(U) = R(z(k), k(T))$ and

$$U[e](k', i) = \begin{cases} [T, e](k', i) & \text{if } e \in [1, \mathrm{dep}(U)] \setminus \{s\} \text{ and } i \in [1, \mathrm{len}(U[e])] \\ [T, e](k', i) & \text{if } e = s \text{ and } i \in [1, \mathrm{len}(U[e])] \setminus \{t(1, s)\} \\ z(k') & \text{if } e = s \text{ and } i = t(1, s) \end{cases}$$

and we define $R(k, z, T) \in \mathrm{tins}(k, z)$ by putting $R(k, z, T) = U$ and we remark that $R(k, z, T)$ may be called the *roinsertion* of (k, z, T) ,and we note that clearly

$$U[s](k, t(s)) = x(1, s).$$

Definition 8.3. Given any $w \in \mathrm{popre}(2)$ we put

$$\mathrm{tins}(k, w) = \begin{cases} \mathrm{stab}(2) & \text{if } \mathrm{len}(w) = 0 \\ \mathrm{tins}(k, w[1]) & \text{if } \mathrm{len}(w) \neq 0 \end{cases}$$

and

$$\mathrm{tits}(k, w) = \begin{cases} \mathrm{stab}(2) & \text{if } \mathrm{len}(w) = 0 \\ \mathrm{tins}(k, w[\mathrm{len}(w)]) & \text{if } \mathrm{len}(w) \neq 0 \end{cases}$$

and we remark that a member of tins(k, w) (resp: tits(k, w)) may be called a *tabular insertible* (resp: *tabular terminable*) of (k, w).

Definition 8.4. Given any $w \in \mathrm{leb}(k)$ and $T \in \mathrm{tins}(k, w)$, upon letting $r = \mathrm{len}(w)$, clearly there exists a unique sequence $B = B(i)_{1 \leq i \leq r+1}$, with $B(i) \in \mathrm{tins}(k, w[i])$ for all $i \in [1, r]$ and $B(r+1) \in \mathrm{tits}(k, w)$, such that $B(1) = T$ and $T(i+1) = R(k, w[i], B(i))$ for all $i \in [1, r]$, and we define $\mathrm{RQ}(k, w, T)$ by putting $\mathrm{RQ}(k, w, T) = B$ and we remark that $\mathrm{RQ}(k, w, T)$ may be called the *sequential roinsertion* of (k, w, T), and we define $R(k, w, T) \in \mathrm{tits}(k, w)$ by putting $R(k, w, T) = B(r+1)$ and we remark that $R(k, w, T)$ may be called the *roinsertion* of (k, w, T).

Definition 8.5. Thus we get a map $R_k: \mathrm{leb}(k) \to \mathrm{stab}(2)$ by putting $R_k(w) = R(k, w, \mathrm{stab}_0(2))$ for all $w \in \mathrm{leb}(k)$, and we remark that $R_k(w)$ may be called the *roinsertion* of (k, w).

Lemma 8.6. *Given any $w \in \mathrm{leb}(k)$, upon letting $T = R_k(w)$, we have*

$$k(T) = R(k(w)) \text{ and } \mathrm{con}(T) = \mathrm{con}[w] \text{ and } \mathrm{are}(T) = \mathrm{len}(w).$$

Proof. Obvious. □

Lemma 8.7. *In the situation of* (8.2) *we have*

$$\mathrm{dep}(U) > 0 \text{ and } U[1] = R(k, z, [T, 1]) = \begin{cases} R(k, z, T[i]) & \text{if } d > 0 \\ R(k, z, \mathrm{vec}_0(2)) & \text{if } d = 0 \end{cases}$$

and in the situation of (8.4), *upon letting $d = \mathrm{dep}(T)$, we have*

$$\mathrm{dep}(R(k, w, T)) = 0 \Leftrightarrow r = 0 = d$$

and in the situation of (8.4) *we also have*

$$[R(k, w, T), 1] = R(k, w, [T, 1]).$$

Proof. Obvious. □

Lemma 8.8. *Given any $w \in \mathrm{leb}(k)$, upon letting $r = \mathrm{len}(w)$ and $T = R_k(w)$, we have*

$$\mathrm{dep}(T) = 0 \Leftrightarrow r = 0$$

and we have

$$\mathrm{RL}(k, w) = [T, 1] = \begin{cases} T[1] & \text{if } r > 0 \\ \mathrm{vec}_0(2) & \text{if } r = 0 \end{cases}$$

and we have

$$k([T, 1]) = [R(k(w)), 1] = \mathrm{RL}(k(w)).$$

Proof. Obvious. □

Theorem 8.9. *Given any $w \in \mathrm{leb}(k)$, upon letting $T = R_k(w)$, we have*

$$k([T, 1]) = [R(k(w)), 1] = \mathrm{veg}(k(w)) = k(\mathrm{veg}(w))$$

and

$$[T, 1] = \mathrm{veg}(w) \text{ and } \mathrm{len}([T, 1]) = \mathrm{inc}(w) = \mathrm{inc}(k(w)).$$

Proof. Follows from (7.8) and (8.8). □

Definition 8.10. Given any $m \in N^*(2)$, we put

$$\mathrm{leb}(k, m) = \{w \in \mathrm{leb}(k) \colon w \leqslant m\}$$

Lemma 8.11. *For every* $m \in N^*(2)$ *and* $w \in \text{leb}(k, m)$, *we have* $R_k(w) \in \text{stab}(2, m)$.

Proof. Follows from (8.6). □

Definition 8.12. Given any $m \in N^*(2)$, in view of (8.11) we get a map $R_{k,m}$: $\text{leb}(k, m) \to \text{stab}(2, m)$ by putting $R_{k,m}(w) = R_k(w)$ for all $w \in \text{leb}(k, m)$.

Theorem 8.13. *Given any* $m \in N^*(2)$ *and* $w \in \text{leb}(k, m)$, *upon letting* $T = R_{k,m}(w)$, *we have*

$$k(T) = R(k(w)) \textit{ and } \text{con}(T) = \text{con}[w] \textit{ and } \text{are}(T) = \text{len}(w)$$

and

$$k([T, 1]) = [R(k(w)), 1] = \text{veg}(k(w)) = k(\text{veg}(w))$$

and

$$[T, 1] = \text{veg}(w) \textit{ and } \text{len}([T, 1]) = \text{inc}(w) = \text{inc}(k(w)).$$

Proof. Follows from (8.6) and (8.9). □

Definition 8.14. Given any $m \in N^*(2)$, we note that for each $t \in \text{mon}(2, m)$ there exists a unique element $\text{les}[k, m](t)$ in $\text{leb}(k, m)$ such that

$$\text{mos}[\text{les}[k, m](t), m] = t$$

and we remark that $\text{les}[k, m](t)$ may be called the *lexical associate* of (k, m, t), and we note that this gives the *bijective map*

$$\text{les}[k, m]: \text{mon}(2, m) \to \text{leb}(k, m).$$

Lemma 8.15. *For every* $m \in N^*(2)$ *and* $t \in \text{mon}(2, m)$, *we have*

$$\text{abs}(t) = \text{len}(\text{les}[k, m](t)) \textit{ and } \text{veg}(t) = \text{veg}(\text{les}[k, m](t)).$$

Proof. Obvious. □

Definition 8.16. Given any $m \in N^*(2)$, in view of (8.14) we get a map $\text{MR}_{k,m}$: $\text{mon}(2, m) \to \text{stab}(2, m)$ by putting $\text{MR}_{k,m}(t) = R_{k,m}(\text{les}[k, m](t))$ for all $t \in \text{mon}(2, m)$, and we remark that $\text{MR}_{k,m}(t)$ may be called the *monomial roinsertion* of (k, m, t).

Theorem 8.17. *Given any* $m \in N^*(2)$ *and* $t \in \text{mon}(2, m)$, *upon letting* $w = \text{les}[k, m](t)$ *and* $T = \text{MR}_{k,m}(t)$, *we have*

$$k(T) = R(k(w)) \textit{ and } \text{con}(T) = \text{con}[w] \textit{ and } \text{are}(T) = \text{len}(w)$$

and

$$k([T, 1]) = [R(k(w)), 1] = \text{veg}(k(w)) = k(\text{veg}(w))$$

and

$$[T, 1] = \mathrm{veg}(t) = \mathrm{veg}(w)$$

and

$$\mathrm{len}([T, 1]) = \mathrm{ind}(\mathrm{supp}(t)) = \mathrm{inc}(w) = \mathrm{inc}(k(w)).$$

Proof. Follows from (2.7), (8.13) and (8.15). □

Theorem 8.18. *Given any* $m \in N^*(2)$ *and* $t \in \mathrm{mon}(2, m)$ *and* $V \in N$, *upon letting* $T = \mathrm{MR}_{k,m}(t)$, *we have that*

$$t \in \mathrm{mon}[[2, m, V]] \Leftrightarrow T \in \mathrm{stib}(2, m, V).$$

Proof. Follows from (8.17). □

Theorem 8.19. *Given any* $m \in N^*(2)$ *and* $t \in \mathrm{mon}(2, m)$ *and* $p \in N$, *upon letting* $T = \mathrm{MR}_{k,m}(t)$, *we have that*

$$t \in \mathrm{mon}(2, m, p) \Leftrightarrow T \in \mathrm{stab}(2, m, p)$$

and for any $V \in N$ *we have that*

$$t \in \mathrm{mon}[2, m, p, V] \Leftrightarrow T \in \mathrm{stab}[2, m, p, V].$$

Proof. Follows from (8.17). □

Theorem 8.20. *Given any* $m \in N^*(2)$ *and* $t \in \mathrm{mon}(2, m)$ *and* $p \in N$ *and* $a \in \mathrm{vec}(2, m, p)$, *upon letting* $T = \mathrm{MR}_{k,m}(t)$, *we have that*

$$t \in \mathrm{mon}(2, m, p, a) \Leftrightarrow T \in \mathrm{stab}(2, m, p, a)$$

and for any $V \in N$ *we have that*

$$t \in \mathrm{mon}(2, m, p, a, V) \Leftrightarrow T \in \mathrm{stab}(2, m, p, a, V).$$

Proof. Follows from (2.8) and (8.17). □

9. Rodeletion from bitableaux

Throughout this section let there be fixed any $k \in [1, 2]$, and let $k' = 1$ or 2 according as $k = 2$ or 1.

Definition 9.1. Given any $\hat{T} \in \mathrm{stab}(2) \setminus \{\mathrm{stab}_0(2)\}$, upon letting $\hat{d} = \mathrm{dep}(\hat{T})$, firstly we get a unique $\hat{s} \in [1, \hat{d}]$ by putting

$$\hat{s} = \max\{e \in [1, \hat{d}]: \text{for all } f \in [1, \hat{d}] \text{ we have } \hat{T}[e](k', \mathrm{len}(\hat{T}[e])) \geq \hat{T}[f](k', \mathrm{len}(\hat{T}[f]))\}$$

and we note that clearly

$$\hat{s} \in \text{rope}(k'(\hat{T})) = \text{rope}(k(\hat{T}))$$

and upon letting

$$\hat{x} = \text{RDE}(\hat{s}, k(\hat{T})) \text{ and } \hat{t} = \text{RDP}(\hat{s}, \hat{T})$$

we get a unique $\hat{x}_0 \in N^*(2)$ and $\hat{t}_0 \in N^*$ by putting

$$\hat{t}_0 = \hat{t}(1, \hat{s}), \text{ i.e. } \hat{t}_0 = \text{len}(\hat{T}[\hat{s}])$$

and

$$\hat{x}_0(k') = \hat{T}[\hat{s}](k', \hat{t}_0) \text{ and } \hat{x}_0(k) = \hat{x}(1, 1)$$

and we define $\text{ROG}(k, \hat{T}) \in N^*$, $\text{ROE}(k, \hat{T}) \in N^*(2)$ and $\text{ROP}(k, \hat{T}) \in N^*$ by putting $\text{ROG}(k, \hat{T}) = \hat{s}$, $\text{ROE}(k, \hat{T}) = \hat{x}_0$ and $\text{ROP}(k, \hat{T}) = \hat{t}_0$, and we remark that $\text{ROG}(k, \hat{T})$, $\text{ROE}(k, \hat{T})$, and $\text{ROP}(k, \hat{T})$ may respectively be called the *onestep rodeletive tag* of $(k, \hat{T})$, the *onestep rodeletive entry* of $(k, \hat{T})$ and the *onestep rodeletive place* of $(k, \hat{T})$, and now secondly it is clear that there exists a unique $\hat{U} \in \text{stab}(2)$ such that $k(\hat{U}) = \text{RD}(\hat{s}, k(\hat{T}))$ and $\hat{U}[e](k', i) = \hat{T}[e](k', i)$ for all $e \in [1, \text{dep}(\hat{U})]$ and $i \in [1, \text{len}(\hat{U}[e])]$, and we define $\text{RO}(k, \hat{T}) \in \text{stab}_0(2)$ by putting $\text{RO}(k, \hat{T}) = \hat{U}$, and we remark that $\text{RO}(k, \hat{T})$ may be called *onestep rodeletion* of $(k, \hat{T})$.

Definition 9.2. Given any $T \in \text{stab}(2)$, upon letting $V = \text{are}(T)$, firstly it is clear that there exists a unique sequence $B = B(i)_{0 \leq i \leq V}$, with $B(i) \in \text{stib}[2, i]$ for all $i \in [0, V]$, such that $B(V) = T$ and $B(i-1) = \text{RO}(k, B(i))$ for all $i \in [1, V]$, and we define $\text{RDQ}(k, T)$ by putting $\text{RDQ}(k, T) = B$, and we remark that $\text{RDQ}(k, T)$ may be called the *sequential rodeletion* of (k, T), and secondly it is clear that there exists a unique $w \in \text{popre}[2, V]$ such that $w[i] = \text{RDE}(k, B(i))$ for all $i \in [1, V]$, and thirdly by (6.4.3) we see that $w \in \text{leb}[k, V]$, and we define $\text{RD}_k(T) \in \text{leb}(k)$ by putting $\text{RD}_k(T) = w$.

Definition 9.3. Thus we get a map $\text{RD}_k : \text{stab}(2) \to \text{leb}(k)$ which to every $T \in \text{stab}(2)$ associates $\text{RD}_k(T) \in \text{leb}(k)$, and we remark that $\text{RD}_k(T)$ may be called the *rodeletion* of (k, T).

Lemma 9.4. *For every* $T \in \text{stab}(2)$, *upon letting* $w = \text{RD}_k(T)$, *we have* $\text{con}(T) = \text{con}[w]$ *and* $\text{are}(T) = \text{len}(w)$.

Proof. Obvious. □

Lemma 9.5. *In the situation of* (8.2), *upon letting* $\hat{T} = U$ *we have* $\hat{T} \in \text{stab}(2) \setminus \{\text{stab}_0(2)\}$, *and upon letting* $\hat{s}$, $\hat{x}$, $\hat{t}$, $\hat{x}_0$, $\hat{t}_0$, $\hat{U}$ *to be as in* (9.1) *we have*: $\hat{s} = s$, $\hat{x} = x$, $\hat{t} = t$, $\hat{x}_0 = z$, $\hat{t}_0 = t(1, s)$ *and* $\hat{U} = T$.

Proof. Follows from (6.2). □

Lemma 9.6. *In the situation of* (9.1), *upon letting* $T = \hat{U}$ *and* $z = \hat{x}_0$ *we have* $z \in N^*(2)$ *and* $T \in \text{pretins}(k, z)$, *and upon letting* s, x, t, U *to be as in* (8.2) *we have*: $\hat{s} = s$, $x = \hat{x}$, $t = \hat{t}$, $z = \hat{x}_0$, $t(1, s) = \hat{t}_0$ *and* $U = \hat{T}$.

Proof. Follows from (6.3). □

Theorem 9.7. *The maps* R_k: $\text{leb}(k) \to \text{stab}(2)$ *and* RD_k: $\text{stab}(2) \to \text{leb}(k)$ *are inverses of each other, i.e. for every* $w \in \text{leb}(k)$ *we have* $\text{RD}_k(R_k(w)) = w$, *and for every* $T \in \text{stab}(2)$ *we have* $R_k(\text{RD}_k(T)) = T$. *Hence in particular, both the maps* R_k *and* RD_k *are bijective.*

Proof. Follows from (9.5) and (9.6). □

Lemma 9.8. *For every* $m \in N^*(2)$ *and* $T \in \text{stab}(2, m)$, *we have* $\text{RD}_k(T) \in \text{leb}(k, m)$.

Proof. Follows from (9.4). □

Definition 9.9. Given any $m \in N^*(2)$, in view of (9.8) we get a map $\text{RD}_{k,m}$: $\text{stab}(2, m) \to \text{leb}(k, m)$ by putting $\text{RD}_{k,m}(T) = \text{RD}_k(T)$ for all $T \in \text{stab}(2, m)$.

Lemma 9.10. *Given any* $m \in N^*(2)$ *and* $T \in \text{stab}(2, m)$, *upon letting* $w = \text{RD}_{k,m}(T)$, *we have* $\text{con}(T) = \text{con}[w]$ *and* $\text{are}(T) = \text{len}(w)$.

Proof. follows from (9.4). □

Theorem 9.11. *Given any* $m \in N^*(2)$, *we have that the maps* $R_{k,m}$: $\text{leb}(k, m) \to \text{stab}(2, m)$ *and* $\text{RD}_{k,m}$: $\text{stab}(2, m) \to \text{leb}(k, m)$ *are inverses of each other, i.e. for every* $w \in \text{leb}(k, m)$ *we have* $\text{RD}_{k,m}(R_{k,m}(w)) = w$, *and for every* $T = \text{stab}(2, m)$ *we have* $R_{k,m}(\text{RD}_{k,m}(T)) = T$. *Hence in particular, both the maps* $R_{k,m}$ *and* $\text{RD}_{k,m}$ *are bijective.*

Proof. Follows from (8.11), (9.7) and (9.8). □

Definition 9.12. Given any $m \in N^*(2)$, we get a map $\text{MRD}_{k,m}$: $\text{stab}(2, m) \to \text{mon}(2, m)$ by putting $\text{MRD}_{k,m}(T) = \text{mos}[\text{RD}_{k,m}(T), m]$ for all $T \in \text{stab}(2, m)$, and we remark that $\text{MRD}_{k,m}(T)$ may be called the *monomial rodeletion* of (k, m, T).

Theorem 9.13. *Given any* $m \in N^*(2)$, *we have that the maps* $MR_{k,m}$: $\text{mon}(2, m) \to \text{stab}(2, m)$ *and* $\text{MRD}_{k,m}$: $\text{stab}(2, \text{m}) \to \text{mon}(2, m)$ *are inverses of each other, i.e. for every* $t \in \text{mon}(2, m)$ *we have* $\text{MRD}_{k,m}(\text{MR}_{k,m}(t)) =$

t, and for every $T \in \mathrm{stab}(2, m)$ *we have* $\mathrm{MR}_{k,m}(\mathrm{MRD}_{k,m}(T)) = T$. *Hence in particular, both the maps* $\mathrm{MR}_{k,m}$ and $\mathrm{MRD}_{k,m}$ *are bijective.*

Proof. Follows from (8.14) and (9.11). □

Theorem 9.14. *Given any* $m \in N^*(2)$ *and* $V \in N$, *the maps* $\mathrm{MR}_{k,m}$ *and* $\mathrm{MRD}_{k,m}$ *induce bijective maps* $\mathrm{mon}[[2, m, V]] \to \mathrm{stib}(2, m, V)$ *and* $\mathrm{stib}(2, m, V) \to \mathrm{mon}[[2, m, V]]$ *which are inverses of each other.*

Proof. Follows from (8.18) and (9.13). □

Theorem 9.15. *Given any* $m \in N^*(2)$ *and* $p \in N$, *the maps* $\mathrm{MR}_{k,m}$ *and* $\mathrm{MRD}_{k,m}$ *induce bijective maps* $\mathrm{mon}(2, m, p) \to \mathrm{stab}(2, m, p)$ *and* $\mathrm{stab}(2, m, p) \to \mathrm{mon}(2, m, p)$ *which are inverses of each other, and for every* $V \in N$, *the maps* $\mathrm{MR}_{k,m}$ *and* $\mathrm{MRD}_{k,m}$ *induce bijective maps* $\mathrm{mon}[2, m, p, V] \to \mathrm{stab}[2, m, p, V]$ *and* $\mathrm{stab}[2, m, p, V] \to \mathrm{mon}[2, m, p, V]$ *which are inverses of each other.*

Proof. Follows from (8.19) and (9.13). □

Theorem 9.16. *Given any* $m \in N^*(2)$ *and* $p \in N$ *and* $a \in \mathrm{vec}(2, m, p)$, *the maps* $\mathrm{MR}_{k,m}$ *and* $\mathrm{MRD}_{k,m}$ *induce bijective maps* $\mathrm{mon}(2, m, p, a) \to \mathrm{stab}(2, m, p, a)$ *and* $\mathrm{stab}(2, m, p, a) \to \mathrm{mon}(2, m, p, a)$ *which are inverses of each other, and for every* $V \in N$, *the maps* $\mathrm{MR}_{k,m}$ *and* $\mathrm{MRD}_{k,m}$ *induce bijective maps* $\mathrm{mon}(2, m, p, a, V) \to \mathrm{stab}(2, m, p, a, V)$ *and* $\mathrm{stab}(2, m, p, a, V) \to \mathrm{mon}(2, m, p, a, V)$ *which are inverses of each other.*

Proof. Follows from (8.20) and (9.13). □

References

[1] S.S. Abhyankar, Combinatorie des tableaux de Young, varietes determinantielles et calcul de fonctions de Hilbert, Rendiconti del Seminario Matematico des Universita et Polytecnico di Torino, 42 (1984) 65–88.
[2] S.S. Abhyankar, Determinantal loci and enumerative combinatorics of Young tableaux, Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata, (1987) 1–26.
[3] S.S. Abhyankar, Enumerative Combinatorics of Young Tableaux (Marcel Dekker, New York, 1988).
[4] S.S. Abhyankar and S.R. Ghorpade, Young tableaux and linear independence of standard monomials in multiminors of a multimatrix, forthcoming.
[5] S.S. Abhyankar and D.M. Kulkarni, Coinsertion and standard bitableaux, forthcoming.
[6] C. De Concini, D. Eisenbud and C. Procesi, Young diagrams and determinantal varieties, Inv. Math. 56 (1980) 129–165.
[7] J. Desarmenian, J.P.S. Kung and G.C. Rota, Invariant theory, Young bitableaux, and combinatorics, Advances Mathemat. 27 (1978) 63–92.
[8] P. Doubilet, G.C. Rota and J. Stein, Foundations of combinatorics IX: Combinatorial methods in invariant theory, Studies Applied Mathemat. 53 (1974) 185–216.

[9] D.E. Knuth, Permutations, matrices, and generalized Young tableaux, Pacific J Mathemat. 34 (1970) 709–727.
[10] D.E. Knuth, The Art of Computer Programming, Vol. 3, Sorting and Searching (Addison Wesley, Reading, 1973).
[11] J.P.S. Kung (Ed.), Young Tableaux in Combinatorics, Invariant Theory, and Algebra (Academic Press, New York, 1982).
[12] G. de B. Robinson, On the representations of the symmetric group, American J. Mathemat. 60 (1938) 745–760.
[13] C. Schensted, Longest increasing and decreasing subsequences, Canadian J. Mathemat. 13 (1961) 179–191.